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Advancements in fixed point results for ordered metric space via extended F-weak contraction with applications

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Abstract:

In (Ann. Math. Comp. Sc., 20, 82–97, 2024), Raji and Ibrahim introduced a new class of contractions, termed modified α-F-weak contractions, and established fixed point results in complete metric spaces, thereby extending the classical Banach contraction principle. Building on this, the present paper introduces an extended F-weak contraction mapping in ordered metric spaces and derives corresponding fixed-point results. The theoretical contributions are further illustrated through numerical examples. As an application, we demonstrate the existence and uniqueness of solutions to a first-order ordinary differential equation subject to periodic boundary conditions.

Keywords:

Fixed point, F-contraction, Extended F-weak contraction, α-F-weak contraction, Periodic points.

Mathematics subject Classification 47H10; 54H25.

1 Introduction

Fixed point theory serves as a dynamic link between analysis and topology, providing a robust analytical framework with applications ranging from pure mathematics to practical problems in applied sciences. Its prominence largely stems from the classical Banach contraction principle [3]. Following Banach, Wardowski [1] introduced a significant reformulation through the concept of F-contraction, which employs an auxiliary function to quantify how a mapping deviates from the fixed-point property under certain conditions. Later, Wardowski [9] expanded this idea by introducing F-weak contraction, thereby generalizing Banach's principle in several directions. Subsequent contributions enriched the field: Hussain et al. [19] proposed the notion of α-GF-contraction as a broader generalization of F-contraction, establishing new fixed point results. Gopal et al. [12] developed the concept of α-type F-contractions for self-mappings in complete metric spaces and provided sufficient conditions for the existence and uniqueness of fixed points. Additional generalizations can be found in [4,7,10,13,14,16].

More recently, Raji and Ibrahim [17] introduced a modified α-$\mathcal{F}$-weak contraction for self-mappings in metric spaces and established fixed point theorems in this setting. Building on this line of research, we propose the notion of extended $\mathcal{F}$-weak contraction mappings in ordered-metric spaces and establish existence and uniqueness results for solutions of first-order ordinary differential equations with periodic boundary conditions. Furthermore, illustrative examples are presented to demonstrate the applicability of the new theory.

Throughout the article, we denoted $\mathbb{R}$ is the set of all real numbers and $\mathbb{S}$ the collections of all function $\sigma: \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim \inf_{t \to s^+} \sigma(t) > 0$ for all $s \geq 0$.

2 Preliminaries

This section begins with the introduction of the concept of F-contraction along with some related definitions.

Definition 2.1 [12] Let $F: (0, +\infty) \to \mathbb{R}$ be a function satisfying the following conditions:

(F_1) F is strictly increasing; that is, for all $\alpha, \beta \in (0, +\infty)$ with $\alpha < \beta$, we have

$$F(\alpha) < F(\beta).$$

(F_2) For any sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of positive numbers,

$$\lim_{n \to \infty} \alpha_n = 0 \text{ if and only } \lim_{n \to \infty} F(\alpha_n) = -\infty.$$

(F_3) There exists $k \in (0,1)$ such that

$$\lim_{n \to 0^+} \alpha^k F(\alpha) = 0.$$

Then, let F be the family of all functions $\mathcal{F}$ satisfying conditions (F_1)– (F_3).

Example 2.2 [11] The following functions $F: (0, +\infty) \to \mathbb{R}$ belongs to class of F:

(i) $F(\alpha) = \ln \alpha\,, \ \alpha > 0;$

(ii) $F(\alpha) = \frac{-1}{\sqrt{\alpha}}, \ \alpha > 0;$

(iii) $F(\alpha) = \ln \alpha + \alpha, \ \alpha > 0;$

(iv) $F(\alpha) = \ln(\alpha^2 + \alpha)\,, \ \alpha > 0.$

Definition 2.3 [2] Let (X, d) be a metric space. A mapping $T: X \to X$ is said to be an F-contraction on X if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $d(Tx, Ty) > 0$, the following inequality holds:

$$\tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big). \qquad (2.1)$$

Definition 2.4 [11] Let (X, d) be a metric space. A mapping $T: X \to X$ is said to be an F-weak contraction on X if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $d(Tx, Ty) > 0$, the inequality

$$\tau + F\big(d(Tx, Ty)\big) \le F\big(M(x, y)\big) \qquad (2.2)$$

holds, where

$$M(x, y) = max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}$$

Remark 2.5 [11]. Every F-contraction is also an F-weak contraction; however, the converse does not always hold. For more detailed examples, we refer the reader to [11].

Definition 2.6 [8] A mapping $T: X \to X$ is said to be α-admissible if there exists a function $\alpha: X \times X \to \mathbb{R}^+$ such that

$$x, y \in X, \alpha(x, y) \ge 1 \Longrightarrow \alpha(Tx, Ty) \ge 1. \qquad (2.3)$$

Definition 2.7[12] Let (X, d) be a metric space and $T: X \to X$ a mapping. The mapping T is called an α-F-contraction on X if there exist $\tau > 0$, a function $F \in \mathcal{F}$, and a function $\alpha: X \times X \to \{-\infty\} \cup (0, +\infty)$ such that for all $x, y \in X$ satisfying $d(Tx, Ty) > 0$, the following condition holds:

$$\tau + \alpha(x, y)F\big(d(Tx, Ty)\big) \le F\big(d(x, y)\big). \qquad (2.4)$$

Definition 2.8 [17] Let (X, d) be a metric space and $T: X \to X$ be a mapping. A mapping T is said to be a modified α-F-weak contraction on X if there exist $\tau > 0$, a function $F \in \mathcal{F}$, and a function $\alpha: X \times X \to \{-\infty\} \cup (0, +\infty)$ such that for all $x, y \in X$ satisfying $d(Tx, Ty) > 0$, the inequality

$$\tau + \alpha(x, y)F\big(d(Tx, Ty)\big) \le F\big(M(x, y)\big), \qquad (2.5)$$

holds, where

$$M(x, y) = max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{d(x, y) + d(x, Ty) + d(y, Tx)}, \frac{d(x, Tx)d(x, Ty) + d(y, Tx)d(y, Ty)}{d(y, Tx) + d(x, Ty)}\right\}$$

Remark 2.9 [18] Every α-F-contraction can be regarded as a modified α-F-weak contraction, however, the reverse implication does not always hold.

Definition 2.10 [6,8] A metric space (X, d) with partial order $\preccurlyeq$ is said to ordered metric space and is denoted by $(X, d, \preccurlyeq)$. For arbitrary $x, y \in X$ and self mapping T on X, we say that

(i) x, y are comparable if either $x \preccurlyeq y$ or $y \preccurlyeq x$.

(ii) T is increasing if $Tx \preccurlyeq Ty$ whenever $x \preccurlyeq y$.

(iii) (X, d) is T-orbitally complete if every Cauchy sequence $\{T^n x\}$ converges in X.

(iv) X is regular if for every increasing sequence sequence $\{x_n\}$ in X with $x_n \to x$, then $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$.

Definition 2.11 Let (X, d) be a metric space, $\sigma \in \mathbb{S}$ and $T: X \to X$ be a mapping. A mapping T is said to be an extended F-weak contraction on X if for all $x, y \in X$ satisfying $d(Tx, Ty) > 0$ and $F \in \mathcal{F}$, then

$$\sigma\big(d(x,y)\big) + F\big(d(Tx,Ty)\big) \leq F\big(M(x,y)\big), \tag{2.6}$$

where

$$M(x,y) = max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}, \frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)}\right\}$$

Definition 2.12[5, 15] An ordered metric space $(X, d, \preccurlyeq)$ is called $\preccurlyeq$-regular if for every increasing sequence $\{x_n\}$ in X with $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a positive integer k_0 such that $x_n \preccurlyeq x$ for all $k \geq k_0$.

3 Main Results

To begin this section, we state the following theorem by utilizing an extended F-weak type contraction.

Theorem 3.1 Let $(X, d, \preccurlyeq)$ be an ordered metric space, $\sigma \in \mathbb{S}$ and $T: X \to X$ an extended F-weak contraction. If $d(X, d)$ is T-orbitally complete and $F \in \mathcal{F}$ such that the following conditions:

(i) there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$,

(ii) T is increasing,

(iii) X is $\preccurlyeq$-regular, as F is continuous.

Then T has a fixed point $x^* \in X$ and for any $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.

ProofLet $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$. We define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If there exists $n \in \mathbb{N} \cup \{0\}$ for which $x_{n+1} = x_n$, then $Tx_n = x_n$. This proves that x_n is a fixed point of T.

Suppose that $x_{n+1} \neq x_n$ that is, $d(Tx_n, Tx_{n+1}) > 0$ for every $n \in \mathbb{N}$. It follows from condition (i) and (ii), that

$$x_0 \preccurlyeq x_1 \preccurlyeq x_2 \ldots \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \cdots. \tag{3.1}$$

On letting $x = x_{n-1}$ and $y = x_n$ in (2.6), we get

$$\sigma\big(d(x_{n-1}, x_n)\big) + F\big(d(Tx_{n-1}, Tx_n)\big) \leq F\big(M(x_{n-1}, x_n)\big),$$

where

$$M(x_{n-1}, x_n) = max\left\{\begin{array}{c} d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \\ \dfrac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{d(x_{n-1}, x_n) + d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}, \\ \dfrac{d(x_{n-1}, Tx_{n-1})d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})d(x_n, Tx_n)}{d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)} \end{array}\right\}$$

implies,

$$\sigma\big(d(x_{n-1}, x_n)\big) + F\big(d(x_n, x_{n+1})\big) \leq F\big(M(x_{n-1}, x_n)\big),$$

where

$$M(x_{n-1},x_n) = max\left\{\begin{matrix} d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), \\ \dfrac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{d(x_{n-1},x_n)+d(x_{n-1},x_{n+1})+d(x_n,x_n)}, \\ \dfrac{d(x_{n-1},x_n)d(x_{n-1},x_{n+1})+d(x_n,x_n)d(x_n,x_{n+1})}{d(x_n,x_n)+d(x_{n-1},x_{n+1})} \end{matrix}\right\}$$

$$= max\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), d(x_{n-1},x_n), d(x_{n-1},x_n)\}$$

$$= F(max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}) \tag{3.2}$$

If $d(x_{n-1},x_n) \leq d(x_n,x_{n+1})$ for some $n \in \mathbb{N}$, to get

$$F\big(d(x_n,x_{n+1})\big) \leq F\big(d(x_n,x_{n+1})\big) - \sigma\big(d(x_{n-1},x_n)\big),$$

which is a contradiction, since $\sigma\big(d(x_{n-1},x_n)\big) > 0$. Hence,

$$F\big(d(x_n,x_{n+1})\big) \leq F\big(d(x_{n-1},x_n)\big) - \sigma\big(d(x_{n-1},x_n)\big).$$

By induction, we get for all $n \in \mathbb{N}$

$$F\big(d(x_n,x_{n+1})\big) \leq F\big(d(x_0,x_1)\big) - n\sigma\big(d(x_0,x_1)\big) \tag{3.3}$$

On taking limit as $n \to \infty$ in (3.3), we get

$$F\big(d(x_n,x_{n+1})\big) = -\infty \tag{3.4}$$

Now, consider (3.4) and (F_2), we have

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{3.5}$$

To show that the sequence $\{x_n\}$ is Cauchy. Assume that there exists $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that for all $k \in \mathbb{N}$,

$$n_k > m_k \geq k, d\big(x_{n_k},x_{m_k}\big) \geq \epsilon$$

and

$$d\big(x_{n_k-1},x_{m_k}\big) < \epsilon.$$

Then,

$$\epsilon \leq d\big(x_{n_k},x_{m_k}\big) \leq d\big(x_{n_k},x_{n_k-1}\big) + d\big(x_{n_k-1},x_{m_k}\big) \leq d\big(x_{n_k},x_{n_k-1}\big) + \epsilon,$$

implies,

$$\lim_{k\to\infty} d\big(x_{n_k},x_{m_k}\big) = \epsilon. \tag{3.6}$$

Again, we get

$$\epsilon \leq d\big(x_{n_k},x_{n_k+1}\big) + d\big(x_{n_k+1},x_{m_k+1}\big) + d\big(x_{m_k+1},x_{m_k}\big) \tag{3.7}$$

Letting $k \to \infty$ in (3.7), we have

$$\epsilon \leq \liminf_{k\to\infty} d\big(x_{n_k+1},x_{m_k+1}\big).$$

Similarly, we get

$$\epsilon \leq \liminf_{k\to\infty} d\left(x_{n_k+1}, x_{m_k}\right) \text{ and } \epsilon \leq \liminf_{k\to\infty} d\left(x_{m_k+1}, x_{n_k}\right).$$

Implies there exists $l \in \mathbb{N}$ for all $k \geq l$ with $d\left(x_{n_k+1}, x_{m_k+1}\right) > 0,\ d\left(x_{n_k+1}, x_{m_k}\right) > 0$ and $d\left(x_{m_k+1}, x_{n_k}\right) > 0$. Then, for all $k \geq l$ and setting $x = x_{n_k}$and $y = x_{m_k}$ in (2.6), we have

$$\sigma\left(d\left(x_{n_k}, x_{m_k}\right)\right) + F\left(d\left(Tx_{n_k}, Tx_{m_k}\right)\right) \leq F\left(M\left(x_{n_k}, x_{m_k}\right)\right), \tag{3.8}$$

where

$$M\left(x_{n_k}, x_{m_k}\right) = max\left\{\begin{array}{c} d\left(x_{n_k}, x_{m_k}\right), d\left(x_{n_k}, Tx_{n_k}\right), d\left(x_{m_k}, Tx_{m_k}\right), \\ \dfrac{d\left(x_{n_k}, Tx_{n_k}\right)d\left(x_{m_k}, Tx_{m_k}\right)}{d\left(x_{n_k}, x_{m_k}\right) + d\left(x_{n_k}, Tx_{m_k}\right) + d\left(x_{m_k}, Tx_{n_k}\right)}, \\ \dfrac{d\left(x_{n_k}, Tx_{n_k}\right)d\left(x_{n_k}, Tx_{m_k}\right) + d\left(x_{m_k}, Tx_{n_k}\right)d\left(x_{m_k}, Tx_{m_k}\right)}{d\left(x_{m_k}, Tx_{n_k}\right) + d\left(x_{n_k}, Tx_{m_k}\right)} \end{array}\right\}$$

$$\leq max\left\{\begin{array}{c} d\left(x_{n_k}, x_{m_k}\right), d\left(x_{n_k}, x_{n_k+1}\right), d\left(x_{m_k}, x_{m_k+1}\right), \\ \dfrac{d\left(x_{n_k}, x_{n_k+1}\right)d\left(x_{m_k}, x_{m_k+1}\right)}{d\left(x_{n_k}, x_{m_k}\right) + d\left(x_{n_k}, x_{m_k}\right) + d\left(x_{m_k}, x_{m_k+1}\right) + d\left(x_{m_k}, x_{n_k}\right) + d\left(x_{n_k}, x_{n_k+1}\right)}, \\ \dfrac{d\left(x_{n_k}, x_{n_k+1}\right)\left[d\left(x_{n_k}, x_{m_k}\right) + d\left(x_{m_k}, x_{m_k+1}\right)\right] + \left[d\left(x_{m_k}, x_{n_k}\right) + d\left(x_{n_k}, x_{n_k+1}\right)\right]d\left(x_{m_k}, x_{m_k+1}\right)}{d\left(x_{m_k}, x_{n_k}\right) + d\left(x_{n_k}, x_{n_k+1}\right) + d\left(x_{n_k}, x_{m_k}\right) + d\left(x_{m_k}, x_{m_k+1}\right)} \end{array}\right\}$$

Using the definition of σ, the continuity of F and taking the limit as $k \to \infty$ in (3.8), we get

$$F(\epsilon) < \lim_{k\to\infty} \inf \sigma\left(d\left(x_{m_k}, x_{n_k}\right)\right) + F(\epsilon) \leq F(\epsilon),$$

a contradiction. Implies $\{x_n\}$ is Cauchy sequence with limit $x \in X$.

To show the fixed point of X.suppose $x_n = Tx$ for infinitely many $n \in \mathbb{N}$, then we have the existence of subsequence of $\{x_n\}$ which converges to Tx and the uniqueness of the limit validate the proof. Hence, assume $x_n \neq Tx$ for all $n \in \mathbb{N}_0$. Using the fact that X is $\preccurlyeq$-regular, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with a positive integer k_0 such that $x_{n_k} \preccurlyeq x$ for all $n_k \geq k_0$.

Set $x = x_{n_k}$ and $y = x^*$ for $n_k \geq k_0$ in (2.6), we have

$$\sigma\left(d\left(x_{n_k}, x^*\right)\right) + F\left(d\left(x_{n_k+1}, Tx^*\right)\right) \leq F\left(M\left(x_{n_k}, x^*\right)\right), \tag{3.9}$$

where

$$M\left(x_{n_k}, x^*\right) = max\left\{\begin{array}{c} d\left(x_{n_k}, x^*\right), d\left(x_{n_k}, Tx_{n_k}\right), d(x^*, Tx^*), \dfrac{d\left(x_{n_k}, Tx_{n_k}\right)d(x^*, Tx^*)}{d\left(x_{n_k}, x^*\right) + d\left(x_{n_k}, Tx^*\right) + d\left(x^*, Tx_{n_k}\right)}, \\ \dfrac{d\left(x_{n_k}, Tx_{n_k}\right)d\left(x_{n_k}, Tx^*\right) + d\left(x^*, Tx_{n_k}\right)d(x^*, Tx^*)}{d\left(x^*, Tx_{n_k}\right) + d\left(x_{n_k}, Tx^*\right)} \end{array}\right\}$$

$$= max\left\{\begin{array}{c} d(x_{n_k},x^*), d(x_{n_k},x_{n_k+1}), d(x^*,Tx^*), \dfrac{d(x_{n_k},x_{n_k+1})d(x^*,Tx^*)}{d(x_{n_k},x^*)+d(x_{n_k},Tx^*)+d(x^*,x_{n_k+1})}, \\ \dfrac{d(x_{n_k},x_{n_k+1})d(x_{n_k},Tx^*)+d(x^*,x_{n_k+1})d(x^*,Tx^*)}{d(x^*,x_{n_k+1})+d(x_{n_k},Tx^*)} \end{array}\right\}$$

Letting $n \to \infty$ in (3.9), we get

$$0 < \lim \inf_{d(x_n,x^*)\to 0^+} \sigma\big(d(x_n,x^*)\big) + F\big(d(x^*,Tx^*)\big) \le F\big(d(x^*,Tx^*)\big),$$

a contradiction. Thus, $d(x^*,Tx^*) = 0$, that is, $Tx^* = x^*$. Hence, x^* is the fixed point of T.

The following result is considered by omitting the continuity hypothesis of F and replaced with the continuity of T as,

Theorem 3.2 Let $(X,d,\preccurlyeq)$ be an ordered metric space, $\sigma \in \mathbb{S}$ and $T: X \to X$ an extended F-weak contraction. If $d(X,d)$ is T-orbitally complete and $F \in \mathcal{F}$ such that the following conditions:

(i) there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$,

(ii) T is increasing,

(iii) X is $\preccurlyeq$-regular, as T is continuous.

Then T has a fixed point $x^* \in X$ and for any $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.

Proof The proof is same as Theorem 3.1 to (3.5),

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

From (F_3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty}\Big(\big(d(x_n,x_{n+1})\big)^k F\big(d(x_n,x_{n+1})\big)\Big) = 0. \tag{3.10}$$

Again, from (3.3), for all $n \in \mathbb{N}$, we have

$$\big(d(x_n,x_{n+1})\big)^k\big[F\big(d(x_n,x_{n+1})\big) - F\big(d(x_0,x_1)\big)\big]$$
$$\le -\big(d(x_n,x_{n+1})\big)^k n\sigma d(x_0,x_1) \le 0. \tag{3.11}$$

Using (3.5), (3.9) and letting the limit as $n \to \infty$ in (3.10), we get

$$\lim_{n\to\infty}\Big(n\sigma d(x_0,x_1)\big(d(x_n,x_{n+1})\big)^k\Big) = 0. \tag{3.12}$$

Hence, there exists $n_1 \in \mathbb{N}$ such that $n\big(d(x_n,x_{n+1})\big)^k \le 1$ for all $n \ge n_1$, so that,

$$d(x_n,x_{n+1}) \le \frac{1}{n^{1/k}} \text{ for all } n \ge n_1. \tag{3.13}$$

To show the existence of Cauchy sequence, we consider $m, n \in \mathbb{N}_0$ with $m > n > n_1$, by using (3.11) and the triangle inequality, we get

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + \cdots + d(x_m, x_{m+1})$$
$$< \sum_{i=1}^{\infty} d(x_i, x_{i+1}) \leq \sum_{i=1}^{\infty} \frac{1}{i^{1/k}}. \tag{3.14}$$

As the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent, letting the limit as $m, n \to \infty$, to get

$$\lim_{m,n\to\infty} d(x_n, x_m) = 0. \tag{3.15}$$

Which proves that $\{x_n\}$ is Cauchy sequence in X.

Since X is T-orbitally complete, there exists $x^* \in X$ such that $\lim_{n\to+\infty} x_n = x^*$. Then, the continuity of T implies

$$x^* = \lim_{n\to\infty} x_{n+1}$$
$$= T\left(\lim_{n\to\infty} x_n\right)$$
$$= Tx^*. \tag{3.15}$$

Thus, $Tx^* = x^*$. Hence, x^* is the fixed point of T.

Example 3.3 Let $X = [0,2]$ with the usual metric $d(u, v) = |u - v|$. Define the partial order $\preccurlyeq$ by

$$x \preccurlyeq y \Leftrightarrow x \leq y, \text{and } U = [0,1], V = \left(1, \tfrac{3}{2}\right] \text{and } W = \left(\tfrac{3}{2}, 2\right].$$

Let $F \in \mathcal{F}$ be define by $F(s) = -\frac{1}{\sqrt{s}}\,(s > 0)$ and $\sigma(t) = \frac{1}{4}\,(t \geq 0)$. Define $T: X \to X$ by

$$T(x) = \begin{cases} \frac{9}{5}, & for\ x \in U; \\ 2, & for\ x \in V; \\ 2 & for\ x \in W. \end{cases}$$

Then T is $\preccurlyeq$-increasing. We consider the two cases to validate the extended F-weak contraction on X in inequality (2.6) for every pair (x, y) with $d(Tx, Ty) > 0$.

Case 1: With $x \in U\ and\ y \in V$, we have

$$d(Tx, Ty) = \left|\frac{9}{5} - 2\right| = \frac{1}{5}.$$

Thus

$$F\big(d(Tx, Ty)\big) = F\left(\tfrac{1}{5}\right) = -\frac{1}{\sqrt{1/5}} = -\sqrt{5}\ ,\ \sigma\big(d(x, y)\big) = \tfrac{1}{4},$$

so the left hand side (L.H.S) of (2.6) is

$$\sigma\big(d(x,y)\big) + F\big(d(Tx,Ty)\big) = \frac{1}{4} + F\left(\frac{1}{5}\right) = \frac{1}{4} - \sqrt{5}\,.$$

The right hand side R.H.S, $M(x,y)$ with $x \in U \; and \; y \in V$, we have

$$d(x,y) = y - x$$

$$d(x,Ty) = |x-2| = 2 - x$$

$$d(x,Tx) = \left|x - \frac{9}{5}\right| = \frac{9}{5} - x$$

$$d(y,Ty) = |y-2| = 2 - y$$

$$d(y,Tx) = \left|y - \frac{9}{5}\right| = \frac{9}{5} - y.$$

Hence, for every $x \in U \; and \; y \in V$, we have

$$d(x,Tx) = \frac{9}{5} - 1 = \frac{4}{5}, \qquad d(y,Ty) = 2 - \frac{3}{2} = \frac{1}{2},$$

$$d(x,Ty) = 2 - 1 = 1, \qquad d(y,Tx) = \frac{9}{5} - \frac{3}{2} = \frac{3}{10},$$

and $d(x,y) \geq 0$. Therefore,

$$M(x,y) = max \left\{ \begin{array}{c} d(x,y), d(x,Tx), d(y,Ty), \dfrac{d(x,Tx)d(y,Ty)}{d(x,y) + d(x,Ty) + d(y,Tx)}, \\ \dfrac{d(x,Tx)d(x,Ty) + d(y,Tx)d(y,Ty)}{d(y,Tx) + d(x,Ty)} \end{array} \right\} = \frac{4}{5}.$$

Consequently,

$$F\big(M(x,y)\big) \leq F\left(\frac{4}{5}\right) = -\frac{1}{\sqrt{4/5}} = -\frac{\sqrt{5}}{2}.$$

To compare the L.H.S and R.H.S. We need to show

$$\frac{1}{4} - \sqrt{5} \leq -\frac{\sqrt{5}}{2},$$

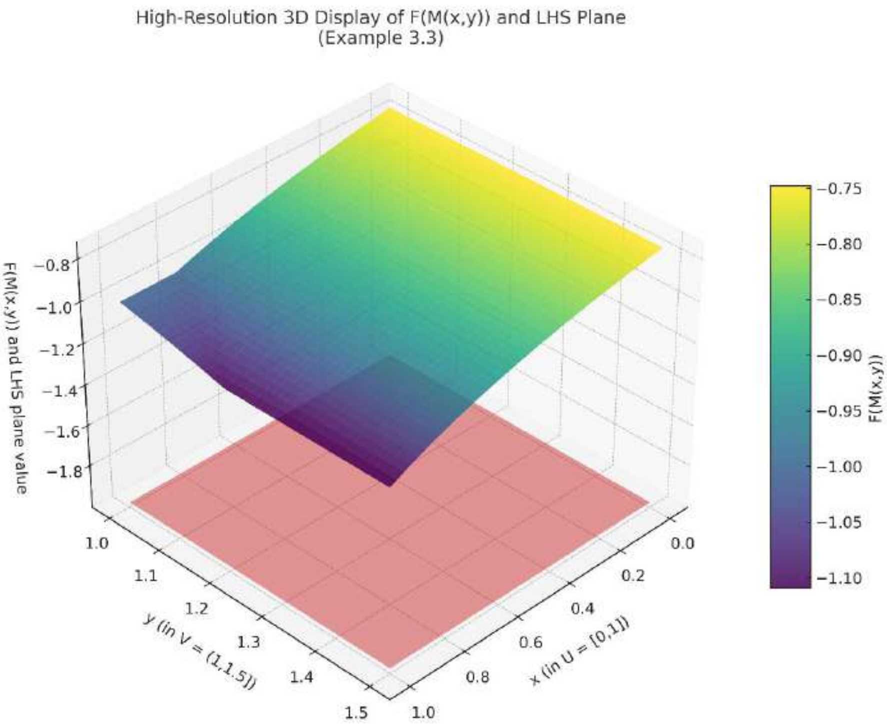
Equivalent to

$$\frac{\sqrt{5}}{2} \leq \sqrt{5} - \frac{1}{4}.$$

Therefore, for every $x \in U \; and \; y \in V$, we have

$$\sigma\big(d(x,y)\big) + F\big(d(Tx,Ty)\big) \leq F\big(M(x,y)\big)$$

Case 2: Again with $x \in V \; and \; y \in W$. For these pairs $T(x) = T(y) = 2$, so $d(Tx,Ty) = 0$ and the hypothesis of (2.6) which is require when $d(Tx,Ty) > 0$ does not apply. Thus, T satisfies (2.6) for every pair with $d(Tx,Ty) > 0$, that is, T is an extended F-weak contraction on X. By Theorem 3.1 (or Theorem 3.2 under the continuity hypothesis), T has a fixed point $x^* \in X$ and for any $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.



A 3-D display that visualizes the surface $z = F(M(x,y))$ over $x \in U = [0,1]$ and $y \in V = (1,3/2]$, together with the constant LHS plane $o(d(x,y)) + F(d(Tx,Ty))$, which is constant because $d(Tx,Ty) = 1/5$ for $x \in U, y \in V$.

The plot shows the lower flat plane is the LHS value $1/4 + F(1/5)$ and the curved surface above it is $F(M(x,y))$. Wherever the surface is above the plane the inequality

$$o(d(x,y)) + F(d(Tx,Ty)) \leq F(M(x,y))$$

holds (as required for the extended F-weak contraction).

Next, to prove the existence of uniqueness result, we will consider the following hypothesis of the Theorem:

S: $\quad Fix(T) \coloneqq \{x \in X, Tx = x\}$ is a totally ordered set

Theorem 3.4 Adding condition (S) to the hypothesis of Theorem 3.1 (respectively Theorem 3.2), then T has a uniqueness of fixed point.

Proof Suppose that y^* is another fixed point $Fix(T)$, that is $d(x^*, Ty^*) = d(x^*, y^*) > 0$. Then from (2.6), we get

$$\sigma\big(d(x^*, y^*)\big) + F\big(d(x^*, y^*)\big) \leq \sigma\big(d(x^*, y^*)\big) + F\big(d(Tx^*, Ty^*)\big),$$

$$\leq F\left(max\begin{Bmatrix} d(x^*,y^*), d(x^*,Tx^*), d(y^*,Ty^*), \frac{d(x^*,Tx^*)d(y^*,Ty^*)}{d(x^*,y^*)+d(x^*,Ty^*)+d(y^*,Tx^*)}, \\ \frac{d(x^*,Tx^*)d(x^*,Ty^*)+d(y^*,Tx^*)d(y^*,Ty^*)}{d(y^*,Tx^*)+d(x^*,Ty^*)} \end{Bmatrix}\right)$$

$$= F\big(d(x^*,y^*)\big),$$

a contradiction. Implies $d(x^*,y^*) = 0$, that is, $x^* = y^*$.

4 Application

As an application of our results, we establish the existence of solution of first order periodic boundary value problem with respect to its upper or lower solution.

$$\begin{cases} g'(s) = T\big(s,g(s)\big), \quad s \in I = [0,S] \\ g(0) = g(S). \end{cases} \tag{4.1}$$

where $T: t \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $S > 0$.

Suppose $\mathcal{C}(I)$ is the space of all continuous functions define on I. In the sequel, we begin with the following definitions.

Definition 4.1 [17] A function $\mu \in \mathcal{C}'(I)$ is referred to as a lower solution of (4.1), if

$$\begin{cases} \mu'(s) \leq T\big(s,\mu(s)\big), x \in I \\ \mu(0) \leq \mu(S). \end{cases}$$

Definition 4.2 [17] A function $\mu \in \mathcal{C}'(I)$ is referred to as an upper solution of (4.1), if

$$\begin{cases} \mu'(s) \geq T\big(s,\mu(s)\big), x \in I \\ \mu(0) \geq \mu(S). \end{cases}$$

Theorem 4.3 In view of condition (4.1). Suppose the following hold:

(i) for all $x, y \in \mathbb{R}$ with $x \leq y$, there exists $\tau > 0$ such that

$$0 \leq T(s,y) + e^{-\tau}y - [T(s,x) + e^{-\tau}x] \leq e^{-\tau}(y-x)$$

(ii) for all $x, y \in \mathbb{R}$ and for all $s \in I$ with $w(a,b) \geq 0$, there exists a function $w: \mathbb{R}^2 \to \mathbb{R}$ such that

$$w\left(\int_0^S H(s,t)\big|T\big(t,g(t)\big) + e^{-\tau}g(t)\big|dt, \mu(s)\right) \geq 0,$$

where $\mu \in \mathcal{C}'(I)$ is the lower solution of (4.1).

(iii) there exists $w\big(x(s),y(s)\big) \geq 0$, for all $s \in I$ and $x, y \in \mathcal{C}'(I)$ such that

$$w\left(\int_0^S H(s,t)\big|T\big(t,x(t)\big) + e^{-\tau}x(t)\big|dt, \quad \int_0^S H(s,t)\big|T\big(t,y(t)\big) + e^{-\tau}y(t)\big|dt\right) \geq 0$$

(iv) for all $n \in \mathbb{N}$, $x_n \to x \in \mathcal{C}'(I)$ and $w(x_{n+1},x_n) \geq 0$ implies $w(x_n,x) \geq 0$.

Then, the existence of a lower solution of (4.1), thereby ensures the existence and uniqueness of a solution of (4.1).

Proof The condition (4.1) can be rewritten as

$$\begin{cases} g'(s) + e^{-\tau} g(s) = T(s, g(s)) + e^{-\tau} g(s) \text{ for all } s \in I \\ \qquad g(0) = g(S) \end{cases} \tag{4.2}$$

with the integral equation form as

$$g(s) = \int_0^S H(s,t)\left[T(t, g(t)) + e^{-\tau} g(t)\right] dt \tag{4.3}$$

Let $\alpha = e^{-\tau}$. The integrating factor of $g'(s) + \alpha g(s) = f(s)$ is $e^{\alpha s}$. Solving with periodic boundary condition $g(0) = g(S)$ gives the Green's function

$$H(s,t) = \begin{cases} \dfrac{e^{\alpha(S+t-s)}}{e^{\alpha S} - 1}, & 0 \le t < s \le S, \\ \dfrac{e^{\alpha(t-s)}}{e^{\alpha S} - 1}, & 0 \le s < t \le S. \end{cases}$$

The function $X: \mathcal{C}(I) \to \mathcal{C}(I)$ is defined for all $s \in I$ as

$$(Xg)(s) = \int_0^S H(s,t)\left[T(t, g(t)) + e^{-\tau} g(t)\right] dt. \tag{4.4}$$

If $g \in \mathcal{C}(I)$ is a fixed point of X, then $g \in \mathcal{C}'(I)$ is a solution of (4.3), implies the solution of (4.1).

The metric d on $\mathcal{C}(I)$ is defined for all $g, h \in \mathcal{C}(I)$ as

$$d(g,h) = \sup_{s \in I} |g(s) - h(s)|. \tag{4.5}$$

By $\mathcal{C}(I)$, we define a partial order $\preccurlyeq$ for all $s \in \mathcal{C}(I)$ as

$$g, h \in \mathcal{C}(I): g \preccurlyeq h \Leftrightarrow g(s) \le h(s) \tag{4.6}$$

$(\mathcal{C}(I), d, \preccurlyeq)$ is clearly a complete ordered metric space. To verify other conditions of Theorem 3.1, we consider the following cases:

Case 1: suppose $\mu \in \mathcal{C}'(I)$ is a lower solution of (4.1) for all $s \in \mathcal{C}(I)$, then

$$\mu'(s) + e^{-\tau}\mu(s) \le T(s, \mu(s)) + e^{-\tau}\mu(s).$$

By multiplying through by $e^{e^{-\tau}s}$ for all $s \in \mathcal{C}(I)$, we get

$$\left(\mu(s) e^{e^{-\tau}s}\right)^t \le \left[T(s, \mu(s)) + e^{-\tau}\mu(s)\right] e^{e^{-\tau}s}$$

so that for all $s \in \mathcal{C}(I)$,

$$\mu(s) e^{e^{-\tau}s} \le \mu(0) + \int_0^s \left[T(s, \mu(s)) + e^{-\tau}\mu(s)\right] e^{e^{-\tau}s} dt.$$

$$(4.7)$$

Since $\mu(0) \le \mu(S)$,

$$\mu(0)e^{e^{-\tau}S} \le \mu(S)e^{e^{-\tau}S} \le \mu(0) + \int_0^S \left[T\big(t,\mu(t)\big) + e^{-\tau}\mu(t)\right]e^{e^{-\tau}t}dt$$

then,

$$\mu(0) \le \int_0^S \frac{e^{e^{-\tau}t}}{e^{e^{-\tau}S}-1}\left[T\big(t,\mu(t)\big) + e^{-\tau}\mu(t)\right]dt. \tag{4.8}$$

By using (4.7) and (4.8), we get

$$\mu(s)e^{e^{-\tau}s} \le \int_0^S \frac{e^{e^{-\tau}t}}{e^{e^{-\tau}S}-1}\left[T\big(t,\mu(t)\big) + e^{-\tau}\mu(t)\right]dt + \int_0^s e^{e^{-\tau}t}\left[T\big(t,\mu(t)\big) + e^{-\tau}\mu(t)\right]dt$$

$$\le \int_0^s \frac{e^{e^{-\tau}(S+t)}}{e^{e^{-\tau}S}-1}\left[T\big(t,\mu(t)\big) + e^{-\tau}\mu(t)\right]dt + \int_s^S \frac{e^{e^{-\tau}t}}{e^{e^{-\tau}S}-1}\left[T\big(t,\mu(t)\big) + e^{-\tau}\mu(t)\right]dt$$

implies for all $s \in I$,

$$\mu(s) \le \int_0^s \frac{e^{e^{-\tau}(S+t-s)}}{e^{e^{-\tau}S}-1}\left[T\big(t,\mu(t)\big) + e^{-\tau}\mu(t)\right]dt + \int_s^S \frac{e^{e^{-\tau}(t-s)}}{e^{e^{-\tau}S}-1}\left[T\big(t,\mu(t)\big) + e^{-\tau}\mu(t)\right]dt$$

$$= \int_0^S H(s,t)\left[T\big(t,g(t)\big) + e^{-\tau}g(t)\right]dt$$

$$= (X\mu)(s)$$

which show that $\mu \preccurlyeq X(\mu)$.

Case 2: Let $g, h \in \mathcal{C}(I)$ such that $g \preccurlyeq h$, then by condition (4.2) for all $s \in I$, we get

$$T\big(s,g(s)\big) + e^{-\tau}g(s) \le T\big(s,h(s)\big) + e^{-\tau}h(s) \tag{4.9}$$

By using (4.4), (4.9) with $H(s,t) > 0$ for $(s,t) \in I \times I$,

$$(Xg)(s) = \int_0^S H(s,t)\left[T\big(t,g(t)\big) + e^{-\tau}g(t)\right]dt$$

$$\le \int_0^S H(s,t)\left[T\big(t,h(t)\big) + e^{-\tau}h(t)\right]dt$$

$$= (Xh)(s)$$

for all $s \in I$, by (4.6), we have $X(g) \preccurlyeq X(h)$ which implies X is increasing.

Case 3: Let an increasing sequence $\{g_n\} \in \mathcal{C}(I)$ be such that $g_n \to g \in \mathcal{C}(I)$, then for each $s \in I$, $\{g_n(s)\}$ is a sequence in $\mathbb{R}$ converging to $g(s)$. Thus, for all $s \in I$ and for all $n \in \mathbb{N}$, $g_n(s) \le g(s)$ for all $n \in \mathbb{N}_0$ which implies $\mathcal{C}(I)$ is $\preccurlyeq$-regular.

To show that X is F-contraction for some $F \in \mathcal{F}$. Let $g, h \in \mathcal{C}(I)$ such that $g \preccurlyeq h$. With (4.2), (4.4) and (4.5),

$$d(Xg,Xh) = \sup_{s\in I}|(Xg)(s) - (Xh)(s)| = \sup_{s\in I}\big((Xh)(s) - (Xg)(s)\big)$$

$$\leq \sup_{s\in I}\int_0^s H(s,t)\big[T\big(t,h(t)\big) + e^{-\tau}h(t) - T\big(t,g(t)\big) - e^{-\tau}g(t)\big]dt$$

$$\leq \sup_{s\in I}\int_0^s H(s,t)[h(t) - g(t)]dt$$

$$= e^{-\tau}d(g,h)\sup_{s\in I}\int_0^s H(s,t)dt$$

$$= e^{-\tau}d(g,h)\sup_{s\in I}\frac{1}{e^{e^{-\tau}S}-1}\left(\frac{1}{e^{-\tau}}e^{e^{-\tau}(S+t-s)}\Big]_0^s + \frac{1}{e^{-\tau}}e^{e^{-\tau}(\tau-s)}\Big]_s^S\right)$$

$$= e^{-\tau}d(g,h)\frac{1}{e^{e^{-\tau}S}-1}\left(e^{e^{-\tau}S} - 1\right)$$

$$= e^{-\tau}d(g,h)$$

$$\leq e^{-\tau}max\left\{\begin{matrix} d(x^*,y^*), d(x^*,Tx^*), d(y^*,Ty^*), \dfrac{d(x^*,Tx^*)d(y^*,Ty^*)}{d(x^*,y^*) + d(x^*,Ty^*) + d(y^*,Tx^*)}, \\ \dfrac{d(x^*,Tx^*)d(x^*,Ty^*) + d(y^*,Tx^*)d(y^*,Ty^*)}{d(y^*,Tx^*) + d(x^*,Ty^*)} \end{matrix}\right\}$$

Since for all $g,h \in C(I)$ with $g \leq h$ we have $d(Xg,Xh) \leq e^{-\tau}d(g,h)$ and $e^{-\tau} < 1$, the operator X is a strict contraction in the ordered sense. Choosing $F(s) = \ln(1+s)$ (continuous and increasing on $[0,\infty)$) and $o \equiv 0$ yields

$$F(d(Xg,Xh)) \leq F(e^{-\tau}d(g,h)) \leq F(M(g,h)),$$

so (2.6) holds. Therefore, all hypotheses of Theorem 3.1 are satisfied and X has a unique fixed point in the monotone orbit issued from the lower solution μ. Moreover, the monotone iterates $X^n\mu$ converge to this fixed point, giving the desired periodic solution.

Conclusion

In this paper, we explored the concept of $\boldsymbol{\mathcal{F}}$-metric spaces to establish some fixed point results for an extended F-weak contraction mapping in ordered metric spaces. The theoretical contributions are further illustrated through numerical examples. As an application, we demonstrated the existence and uniqueness of solutions to a first-order ordinary differential equation subject to periodic boundary conditions.

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