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CONDITIONS UNDER WHICH THE NORM OF AN ELEMENTARY OPERATOR OF LENGTH n >2 IS EXPRESSIBLE IN TERMS OF THE NORMS OF ITS COEFFICIENT OPERATORS

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Abstract:

In the current paper we determine the conditions under which the norm of an elementary operator of finite length is expressible in terms of its coefficient operators using Stampfli's maximal numerical range.

Keywords:

Elementary Operator, Stampfli's maximal numerical range.

1. Introduction

The study of elementary operators has been a topic of research in the past. Basic elementary operator, Jordan elementary operator and an operator of finite length have been considered. Condition under which the norm of an elementary operator of length two can be expressed in terms of its coefficient operators has been determined by King'ang'i [7] using maximal numerical range and Kimeu [4] using Stampfli's maximal numerical range. Kawira *et al* [2] determined the norm of an elementary operator of finite length in a C* algebra.

An operator $T : \mathcal{A} \rightarrow \mathcal{A}$ is called an *elementary operator* if T can be expressed in the form; $Tx = \Sigma A_i x B_i$ with A_i and $B_i (1 \leq i \leq l)$ in $\mathcal{A}$ and $l \in \mathbb{N}$.
J.G Stampfli [13] developed the concept of standard numerical range subset to compute the norm of inner derivations and related bounded linear operators on a Hilbert space. His approach is known as the *Stampfli's maximal numerical range.*

2. Elementary operator of length two

King'ang'i [7] employed maximal numerical range to determine the conditions under which the norm of elementary operator of length 2 is expressible in terms of its coefficient operators.

Theorem 2.1[King'ang'i, 2017]

Let E_2 be an elementary operator of length two on $\mathcal{L}(H)$ then,

$\|E_2\| \geq \sup_{\lambda \in W_{S_1}(S_2{}^*S_1)} \left\{ \left\| \|S_1\| T_1 + \frac{\bar{\lambda}}{\|S_1\|} T_2 \right\| \right\}$, where S_i, T_i are fixed elements of $\mathcal{L}(H)$ for $i = 1,2$, where $W_{S_1}(S_2{}^*S_1)$ is the maximal numerical range of $S_2{}^*S_1$ relative to S_1.
Again, King'ang'i [7] gave the following result;

Theorem 2.2 [Kingangi,2018]
Let E_2 be an elementary operator on $B(H)$ and $S_i, T_i \in B(H)$ for $i = 1,2$. If $\|S_i\| \in W_0(S_i)$ and $\|T_i\| \in W_0(T_i)$ for 1,2 then; $\|E_2\| = \sum_{i=1}^{2} \|T_i\| \, \|S_i\|$
Kimeu *et al* [4] used Stampfli's maximal numerical range to show that the norm of an elementary operator of length two is expressible in terms of its coefficient operators if the maximal numerical ranges map to their respective boundaries.

3. Norm of an Elementary Operator of length n

Kawira *et al* [2] determined the norm of an elementary operator of finite length in a C* algebra by proving the following theorem:

Theorem 3.1

Let H be a complex Hilbert space and B(H) the algebra of all bounded linear operators on H. Let E_n be an elementary operator of length n on $B(H)$. I f for an operator $X \in B(H)$ with $\|X\| = 1$ we have $X(f) = f$ for all unit vectors $f \in H$ then $\|E_n\| = \sum_{i=1}^{n} \|A_i\| \|B_i\|, n \in N$.
In the next theorem, we determine the conditions under which the norm of an elementary operator of length n is expressible in terms of its co-efficient operators.

Theorem 3.2

Let H be a complex Hilbert space and A_i, B_i be bounded linear operators on H for $i = 1, 2, 3, \dots n$. Let E_n be an elementary operator of length,
If $\| A_i \| \in W_0(A_i)$ and $\| B_i \| \in W_0(B_i)$ for $i = 1,2,3, \dots n$ then,

$$\| E_n \| = \sum_{i=1}^{n} \| A_i \| \| B_i \|$$

Proof.

Let $\{x_k\}_{k \geq 1}$ be a sequence of unit vectors on H and $x_k \otimes x_k \in B(H)$ be rank one operator on H for a unit vector $x_k \in H$ defined by $x_k \otimes x_k(y) = \langle y, x_k \rangle x_k$, for all $y \in H$. If

If $\| A_i \| \in W_0(A_i)$ and $\| B_i \| \in W_0(B_i)$ for $i = 1,2,3, \dots n$ then there is a sequence $\{x_k\}_{k \geq 1}$ of

unit vectors on H such that $\lim_{k \to \infty} \langle A_i x_k, x_k \rangle = A_i$, $\lim_{k \to \infty} \|A_i x_k\| = \|A_i\|$ for $i = 1,2,3, \dots, n$ and there is a $\{y_k\}_{k \geq 1}$ of unit vectors on H such that $\lim_{k \to \infty} \langle B_i y_k, y_k \rangle = B_i$, $\lim_{k \to \infty} \|B_i y_k\| = \|B_i\|$ for $i = 1,2,3, \dots, n$

For each $k \geq 1$, we have,

$$\|(E_n(x_k \otimes x_k))x_k\| = \left\|\left(\sum_{i=1}^{n} M_{A_iB_i}(\ x_k \otimes x_k)x_k\right)\right\|$$

$$\leq \left\|\sum_{i=1}^{n} M_{A_iB_i}(\ x_k \otimes x_k)\right\| \|x_k\| \leq \left\|\sum_{i=1}^{n} M_{A_iB_i}\right\| \|x_k \otimes x_k\| \leq \left\|\sum_{i=1}^{n} M_{A_iB_i}\right\| \|x_k\|\|x_k\|$$

$$= \left\|\sum_{i=1}^{n} M_{A_iB_i}\right\|$$

Thus, we have, $\left\|\sum_{i=1}^{n} M_{A_iB_i}\right\|^2 \geq \left\|\left(\sum_{i=1}^{n} M_{A_iB_i}(\ x_k \otimes x_k)x_k\right)\right\|^2$ where,

$$\left\|\left(\sum_{i=1}^{n} M_{A_iB_i}(\ x_k \otimes x_k)x_k\right)\right\|^2$$

$$= \|(A_1(x_k \otimes x_k)B_1 + A_2(x_k \otimes x_k)B_2 + A_3(x_k \otimes x_k)B_3 + \cdots + A_n(x_k \otimes x_k)B_n)x_k\|^2$$

$$= \|A_1(x_k \otimes x_k)B_1x_k + A_2(x_k \otimes x_k)B_2x_k + A_3(x_k \otimes x_k)B_3x_k + \cdots A_n(x_k \otimes x_k)B_nx_k\|^2$$

$$= \|\langle B_1x_k, x_k\rangle A_1x_k + \langle B_2x_k, x_k\rangle A_2x_k + \langle B_3x_k, x_k\rangle A_3x_k + \cdots + \langle B_nx_k, x_k\rangle A_nx_k\|^2$$

$$= \langle\langle B_1x_k, x_k\rangle A_1x_k + \langle B_2x_k, x_k\rangle A_2x_k + \langle B_3x_k, x_k\rangle A_3x_k + \cdots + \langle B_nx_k, x_k\rangle A_nx_k, \langle B_1x_k, x_k\rangle A_1x_k + \langle B_2x_k, x_k\rangle A_2x_k + \langle B_3x_k, x_k\rangle A_3x_k + \cdots + \langle B_nx_k, x_k\rangle A_nx_k\rangle$$

$$= \|\langle B_1x_k, x_k\rangle A_1x_k\|^2 + \|\langle B_2x_k, x_k\rangle A_2x_k\|^2 + \|\langle B_3x_k, x_k\rangle A_3x_k\|^2 + \cdots + \|\langle B_nx_k, x_k\rangle A_nx_k\|^2 + 2Re\{\langle\langle B_1x_k, x_k\rangle A_1x_k, \langle B_2x_k, x_k\rangle A_2x_k\rangle + \langle\langle B_1x_k, x_k\rangle A_1x_k, \langle B_3x_k, x_k\rangle A_3x_k\rangle + \cdots \langle\langle B_{n-1}x_k, x_k\rangle A_{n-1}x_k, \langle B_nx_k, x_k\rangle A_nx_k\rangle\}$$

$$= \|A_1x_k\|^2|\langle B_1x_k, x_k\rangle|^2 + \|A_2x_k\|^2|\langle B_2x_k, x_k\rangle|^2 + \|A_3x_k\|^2|\langle B_3x_k, x_k\rangle|^2 + \cdots + \|A_nx_k\|^2|\langle B_nx_k, x_k\rangle|^2 + 2Re\{\langle B_1x_k, x_k\rangle\langle A_1x_k, A_2x_k\rangle\langle B_2x_k, x_k\rangle + \langle B_2x_k, x_k\rangle\langle A_2x_k, A_3x_k\rangle\langle B_3x_k, x_k\rangle + \cdots + \langle B_{n-1}x_k, x_k\rangle\langle A_{n-1}x_k, A_nx_k\rangle\langle B_nx_k, x_k\rangle\}$$

By the CBS inequality, we have;

$$|\langle A_ix_k, A_jx_k\rangle| \leq \|A_ix_k\|\|A_jx_k\| \leq \|A_i\|\|A_j\| \text{ for } i\ j\ =\ 1,2 \ldots n$$

Hence $\lim_{k\to\infty} \langle A_i x_k, A_j x_k \rangle = \|A_i\|\|A_j\|$ for $i\ j\ =\ 1,2 \dots n$

Therefore, taking limits as $k \to \infty$ we obtain,

$\|E_n B(H)\|^2 \geq \|A_1\|^2\|B_1\|^2 + \|A_2\|^2\|B_2\|^2 + \|A_3\|^2\|B_3\|^2 + \dots + \|A_n\|^2\|B_n\|^2 +$

$2\{\|A_1\|\|B_1\|\|A_2\|\|B_2\| + \|A_1\|\|B_1\|\|A_3\|\|B_3\| + \cdots + \|A_1\|\|B_1\|\|A_n\|\|B_n\|\}$

$= \sum_{i=1}^{n}\{\|A_i\|\|B_i\|\}^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\|A_i\|\|B_i\|\|A_j\|\|B_j\|$ for $i \neq j$

$$= \{\|A_1\|\|B_1\| + \|A_2\|\|B_2\| + \|A_3\|\|B_3\| + \cdots + \|A_n\|\|B_n\|\}^2$$

Thus,

$$\|E_n B(H)\|^2 \geq \{\|A_1\|\|B_1\| + \|A_2\|\|B_2\| + \|A_3\|\|B_3\| + \cdots + \|A_n\|\|B_n\|\}^2$$

Or

$$\|E_n B(H)\| \geq \{\|A_1\|\|B_1\| + \|A_2\|\|B_2\| + \|A_3\|\|B_3\| + \cdots + \|A_n\|\|B_n\|\}$$

Therefore,

$$\|E_n B(H)\| \geq \sum_{i=1}^{n} \|A_i\|\|B_i\|$$

Hence,

$$\left\|\sum_{i=1}^{n} M_{A_i B_i}\right\| \geq \sum_{i=1}^{n} \|A_i\|\|B_i\|$$

Since,

$$\left\|\sum_{i=1}^{n} M_{A_i B_i}\right\| \leq \sum_{i=1}^{n} \|A_i\|\|B_i\|$$

holds by the triangular inequality, then we get;

$$\left\|\sum_{i=1}^{n} M_{A_i B_i}\right\| = \sum_{i=1}^{n} \|A_i\|\|B_i\|$$

4. Conclusion

We have determined the conditions under which the norm of an elementary operator of length $n > 2$ is expressible in terms of the norms of its coefficient operators using Stampfli's maximal numerical range. This can also be attempted using a different approach.

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