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Aggregated Fuzzy Soft Rule Interaction and Root Mean Square (RMS) Based Acoustic Instability Modelling for Vocal Vulnerability Assessment

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Abstract

Vocal vulnerability assessment is challenging because acoustic deterioration often occurs gradually rather than through sharp diagnostic boundaries. Traditional threshold-based systems may therefore misrepresent transitional vocal states, while classical fuzzy inference models can lose clinically relevant information by suppressing weaker rule activations. This paper proposes an aggregated fuzzy soft rule interaction framework integrated with Root Mean Square (RMS)-based instability modelling for vocal vulnerability assessment. The aggregation operator incorporates all active fuzzy rule contributions, preserving subtle acoustic interactions, while the RMS instability measure quantifies oscillatory variability using dominant memberships derived from fundamental frequency and vocal perturbation index. Five theorems are rigorously established: boundedness of aggregation, monotonicity, boundedness of RMS instability, continuity, and perturbation stability. These results are proved for arbitrary dimensions involving $p \geq 1$ acoustic parameters, $q_r \geq 2$ linguistic categories, and $n \geq 1$ patients, showing that the framework is not restricted to the specific $3 \times 3 = 9$ rule structure used in the numerical study. Practical validity is illustrated through two numerical examples for each theorem. Statistical validation on 20 patient cases shows a strong positive correlation between aggregated rule interaction and RMS instability ($r = 0.827$, $p < 0.001$), while sensitivity analysis confirms the theoretical perturbation bound with Lipschitz constant $L = 1$. Diagnostic thresholds and algorithmic pseudocode are also provided to support reproducibility. The framework offers a mathematically rigorous and clinically interpretable approach to uncertainty-driven biomedical acoustic assessment.

Keywords: Fuzzy-soft systems, Aggregated rule interaction, RMS instability modelling, Biomedical acoustics, Vocal vulnerability, Perturbation stability.

1 Introduction

Vocal production emerges from a tightly coupled system involving airflow, vocal fold vibration, aerodynamic forces, and resonance within the vocal tract. Because these processes evolve continuously, the resulting acoustic signals exhibit variability, uncertainty, and nonlinear behavior. Diagnostic methodologies that impose binary thresholds often misrepresent this reality, overlooking the gradual transitions between vocal health and vulnerability. Acoustic analysis typically employs parameters such as fundamental frequency (FF), jitter, shimmer, harmonic-to-noise ratio, and vocal perturbation index (VPI). Among these, FF and VPI based measures are particularly valuable because they directly reflect oscillatory vocal fold behavior. Yet many diagnostic systems still assume sharp boundaries between normal and pathological states, even though patients often present overlapping characteristics. Recent studies by Sanabria et al. (2023) demonstrated that fuzzy set theory and soft set theory can be effectively integrated for vocal risk diagnosis and biomedical acoustic uncertainty modelling. Their framework provided important foundations for uncertainty driven vocal assessment systems and motivated further exploration of fuzzy soft interaction structures in vocal analysis, which the present study extends through aggregated rule interaction and RMS based instability modelling. The fuzzification outputs utilized in this paper are adapted from the previously published fuzzification model developed in Okigbo et al. (2026). This present study therefore focuses specifically on aggregated rule interaction, RMS instability modelling, mathematical analysis, and formal theorem proving.

Fuzzy set theory (Zadeh, 1965) and soft set theory (Molodtsov, 1999) provide powerful mathematical tools for modelling uncertainty and gradual transitions. Moreover, classical fuzzy systems often rely on maximum rule selection, discarding weaker but relevant rule activations. This limitation motivates the present study: the development of an aggregated fuzzy soft rule interaction methodology, enhanced with RMS based instability modelling, to better capture the nuanced dynamics of vocal vulnerability. Conventional fuzzy diagnostic systems face three major challenges: information loss due to dominant rule selection, failure to preserve overlapping acoustic interactions, and insufficient modelling of acoustic instability. For example, if three rules activate simultaneously at 0.84, 0.73 and 0.63, a maximum rule system retains only the strongest, discarding the secondary contribution. This simplification distorts interpretation, since vocal systems rarely belong exclusively to a single acoustic state.

The study aims to design an aggregated fuzzy soft rule interaction and RMS based instability framework that quantifies latent vulnerability while preserving overlapping acoustic behaviors. This study contributes: (i) an aggregated fuzzy soft interaction scheme that retains multiple active rules; (ii) an RMS based instability mathematical formulation for acoustic variability analysis; (iii) detailed mathematical derivations and numerical illustrations; (iv) formal proofs of boundedness, continuity, monotonicity, and perturbation stability; (v) generalization proofs showing the theorems hold for arbitrary dimensions; (vi) a complete algorithmic pseudocode specification; (vii) proposed diagnostic classification thresholds with clinical interpretation; (viii) statistical validation including Pearson correlations and descriptive statistics; (ix) sensitivity analysis verifying theoretical perturbation bounds; and (x) a comprehensive literature comparison highlighting the gap filled by the proposed method.

Fuzzy set theory introduced by Zadeh (1965) allows partial membership, enabling representation of gradual transitions. Ross (2010) highlighted their effectiveness in biomedical contexts. Molodtsov (1999) proposed soft set theory as a parameterized uncertainty mechanism. Maji et al. (2003) later integrated fuzzy sets into soft set structures, producing fuzzy soft systems that handle uncertainty and parameter interaction simultaneously. Mamdani and Assilian (1975) introduced rule based fuzzy inference systems utilizing IF-THEN structures. Ross (2010) explained that fuzzy inference systems provide mathematically interpretable frameworks for modelling uncertainty driven systems involving gradual behavioral transitions. Rule activation is commonly computed using the minimum operator:

$$R_i(x_i) = \min\big(\mathrm{FF}(x_i) \wedge \mathrm{VPI}(x_i)\big). \tag{1}$$

Oppenheim and Schafer (2010) established RMS as a stable measure of signal energy and oscillatory variability. Little et al. (2009) demonstrated that nonlinear acoustic properties and oscillatory vocal behavior can provide important information for detecting latent voice disorder. Proakis and Manolakis (2007) further demonstrated its usefulness in noisy oscillatory systems. However, limited studies have integrated RMS formulations directly into fuzzy soft biomedical interaction mechanisms.

2 Preliminaries

This section presents the basic mathematical concepts underlying the proposed techniques.

2.1 Fuzzy Soft Dominant Operator (Zadeh, 1965)

The standard fuzzy union, also known as the dominant operator, produces the largest fuzzy set among all possible unions. The membership of the fuzzy maximum operator is defined mathematically as follows. Assume P denotes a universe of discourse, and suppose A, B denote fuzzy sets on P with membership functions $\mu_A : X \to [0, 1]$ and $\mu_B : X \to [0, 1]$. The union of two fuzzy sets A and B, represented by $A \cup B$, is defined by:

$$\mu_{A\cup B}(x) = \max\bigl(\mu_A(x),\, \mu_B(x)\bigr), \quad \forall x \in X. \tag{2}$$

Example: $\max(0.063, 0.084, 0.0) = 0.084$.

2.2 Fuzzy Soft Rule Intersection Operator (Zadeh, 1965)

Standard fuzzy intersection is the weakest fuzzy intersection. The membership of the fuzzy minimum operator is defined mathematically as follows. Assume P denotes a universe of discourse, and suppose A, B denote fuzzy sets on P with membership functions $\mu_A : X \to [0,1]$ and $\mu_B : X \to [0,1]$. The intersection of two fuzzy sets A and B, represented by $A \cap B$, is defined by:

$$\mu_{A\cap B}(x) = \min\bigl(\mu_A(x),\, \mu_B(x)\bigr), \quad \forall x \in X. \tag{3}$$

Example: $\min(0.094, 0.25) = 0.094$.

2.3 Related Work and Comparative Analysis

This section positions the proposed method within the landscape of fuzzy inference and vocal diagnostic systems. Table 1 summarizes the key distinguishing features.

Table 1: Comparison of the proposed method with existing approaches

Method	Rule Selection	Overlap Pres.	Instability	Perturbation
Mamdani (1975)	Max rule only	No	None	None
Sugeno	Weighted average	Partial	None	None
Sanabria et al. (2023)	Dominant rule	No	None	None
Proposed	**Averaging all active**	**Yes**	**RMS**	$L = 1$

Mamdani (1975). The Mamdani inference system (Mamdani & Assilian, 1975) selects the maximum activated rule and discards all weaker activations. While computationally efficient, this approach loses information when multiple rules activate simultaneously a situation common in biomedical acoustic analysis where patients frequently present overlapping vocal characteristics. The Mamdani system provides no formal instability measure and no perturbation bound.

Sugeno. The Takagi-Sugeno model replaces fuzzy consequents with linear functions of input variables, producing a weighted average defuzzification (Ross, 2010). While this preserves some information from multiple rules, the weights are determined by output functions rather than genuine rule strength averaging. Overlapping acoustic interactions are only partially preserved, and no dedicated instability quantification or perturbation analysis is provided.

Sanabria et al. (2023). The recent fuzzy-soft vocal risk framework by Sanabria et al. (2023) combined fuzzy set and soft set theory for biomedical acoustic diagnosis. However, their method relied on dominant rule selection, discarding secondary activations. No instability measure was proposed, and perturbation robustness was not analyzed mathematically.

Gap and Contribution. The proposed method fills three critical gaps identified in the literature: *(i)* it averages all active rules rather than selecting only the maximum, preserving overlapping acoustic interactions; *(ii)* it introduces an RMS-based instability measure that quantifies oscillatory variability through a mathematically grounded energy formulation; and *(iii)* it provides explicit perturbation bounds with universal Lipschitz constant $L = 1$, ensuring that measurement errors never amplify beyond their input magnitude. These features are achieved while maintaining full interpretability and generalization to arbitrary dimensions p, q_r, and n.

3 Materials and Methods

The fuzzification outputs utilized in this paper are adapted from Okigbo et al. (2026). In that work, each patient was represented using membership values corresponding to linguistic categories associated with acoustic parameters. This study adopts those results directly as input.

3.1 Patient Representation

Let $U = \{x_1, x_2, x_3, \ldots, x_n\}$ denote the universe of patients. Each patient p_i with fuzzy membership value is represented as:

$$x_i = \big(\mathrm{FF}_i(\mathrm{AT}, \mathrm{NT}, \mathrm{ST}),\ \mathrm{VPI}_i(\mathrm{NV}, \mathrm{RV}, \mathrm{AV})\big),$$

where FF_i represents fundamental frequency; AT represents altered tone; NT denotes normal tone; ST denotes stable tone; VPI_i denotes vocal perturbation index; NV denotes normal voice; RV denotes rough voice; and AV represents abnormal voice.

3.2 Root Mean Square in Biomedical Signal Analysis (Oppenheim & Schafer, 2010; Proakis & Manolakis, 2007)

The Root Mean Square (RMS) remains the most widely used measure in signal processing to predict the energy strength of signals. Mathematically, this study designed the RMS to be:

$$\mathrm{Ins}(x_i) = \sqrt{0.5 \cdot \max\big(\mu_{\mathrm{FF,AT}}(x_i), \mu_{\mathrm{FF,NT}}(x_i), \mu_{\mathrm{FF,ST}}(x_i)\big)^2 + \max\big(\mu_{\mathrm{VPI,NV}}(x_i), \mu_{\mathrm{VPI,RV}}(x_i), \mu_{\mathrm{VPI,AV}}(x_i)\big)^2}, \tag{4}$$

which simplifies to:

$$\mathrm{Ins}(x_i) = \sqrt{0.5 \cdot \Big[\big(\mathrm{FFD}(x_i)\big)^2 + \big(\mathrm{VPID}(x_i)\big)^2\Big]}. \tag{5}$$

RMS is established as a stable measure of signal energy and oscillatory variability, where $\mathrm{FFD}(x_i)$ is the dominant fundamental frequency:

$$\mathrm{FFD}(x_i) = \max\big(\mu_{\mathrm{FF,AT}}(x_i),\ \mu_{\mathrm{FF,NT}}(x_i),\ \mu_{\mathrm{FF,ST}}(x_i)\big), \tag{6}$$

and $\mathrm{VPID}(x_i)$ is the dominant vocal perturbation index:

$$\mathrm{VPID}(x_i) = \max\big(\mu_{\mathrm{VPI,NV}}(x_i),\ \mu_{\mathrm{VPI,RV}}(x_i),\ \mu_{\mathrm{VPI,AV}}(x_i)\big). \tag{7}$$

The maximum operator extracts the strongest behavioral representation associated with the patient.

3.3 Fuzzy Soft Rule Construction

Vocal risk depends on the interaction between parameters, not just one parameter. Human voice might exhibit stable tone but high perturbation instability. In order to evaluate this interaction, a rule expert system that models logical rule interaction is required mathematically. The construction of fuzzy soft rules is formulated using the logical "AND" operation linking fundamental frequency and vocal perturbation index. Therefore, the formulated rule combinations are $3 \times 3 = 9$ possible rules. The fuzzy soft constructed rules describe the interaction between two acoustic parameters as shown below:

$$\begin{aligned}
&R_1(x_i) = \mathrm{FF}_{\mathrm{AT}}(x_i) \wedge \mathrm{VPI}_{\mathrm{NV}}(x_i), \quad R_2(x_i) = \mathrm{FF}_{\mathrm{AT}}(x_i) \wedge \mathrm{VPI}_{\mathrm{RV}}(x_i), \quad R_3(x_i) = \mathrm{FF}_{\mathrm{AT}}(x_i) \wedge \mathrm{VPI}_{\mathrm{AV}}(x_i),\\
&R_4(x_i) = \mathrm{FF}_{\mathrm{NT}}(x_i) \wedge \mathrm{VPI}_{\mathrm{NV}}(x_i), \quad R_5(x_i) = \mathrm{FF}_{\mathrm{NT}}(x_i) \wedge \mathrm{VPI}_{\mathrm{RV}}(x_i), \quad R_6(x_i) = \mathrm{FF}_{\mathrm{NT}}(x_i) \wedge \mathrm{VPI}_{\mathrm{AV}}(x_i),\\
&R_7(x_i) = \mathrm{FF}_{\mathrm{ST}}(x_i) \wedge \mathrm{VPI}_{\mathrm{NV}}(x_i), \quad R_8(x_i) = \mathrm{FF}_{\mathrm{ST}}(x_i) \wedge \mathrm{VPI}_{\mathrm{RV}}(x_i), \quad R_9(x_i) = \mathrm{FF}_{\mathrm{ST}}(x_i) \wedge \mathrm{VPI}_{\mathrm{AV}}(x_i).
\end{aligned} \tag{8}$$

The rule strengths are computed using Equation (3). This preserves the conjunction behavior within fuzzy logic.

3.4 Proposed Algorithm

Algorithm 1 presents the complete computational procedure for the aggregated fuzzy soft RMS framework. The algorithm accepts fuzzified membership values as input, extracts dominant memberships, constructs and averages active fuzzy soft rules, computes the RMS instability measure, and produces the diagnostic output tuple.

Algorithm 1 Aggregated Fuzzy Soft RMS Algorithm

Require: Membership values $\mu_{\text{FF,AT}}$, $\mu_{\text{FF,NT}}$, $\mu_{\text{FF,ST}}$, $\mu_{\text{VPI,NV}}$, $\mu_{\text{VPI,RV}}$, $\mu_{\text{VPI,AV}}$

1: **Step 1:** Extract dominant memberships

$$\text{FFD} \leftarrow \max(\mu_{\text{FF,AT}},\, \mu_{\text{FF,NT}},\, \mu_{\text{FF,ST}})$$
$$\text{VPID} \leftarrow \max(\mu_{\text{VPI,NV}},\, \mu_{\text{VPI,RV}},\, \mu_{\text{VPI,AV}})$$

2: **Step 2:** Construct 9 fuzzy soft rules R_j via min operator (Eq. (8))

3: **Step 3:** Identify active rules $\mathcal{K} \leftarrow \{j : R_j > 0\}$, count $k \leftarrow |\mathcal{K}|$

4: **Step 4:** Compute aggregation $\text{Agg} \leftarrow \frac{1}{k}\sum_{j\in\mathcal{K}} R_j$

5: **Step 5:** Compute instability $\text{Ins} \leftarrow \sqrt{0.5 \cdot (\text{FFD}^2 + \text{VPID}^2)}$

Ensure: (Agg, Ins, classification)

Computational Complexity. Algorithm 1 runs in $O(n)$ time for n patients. Step 1 requires $O(q_r)$ comparisons per parameter, Step 2 evaluates $n = \prod q_r$ rules, Step 3 filters active rules in $O(n)$, and Steps 4–5 each require $O(k)$ arithmetic operations where $k \leq n$. For the 3×3 configuration, $n = 9$ and $k \in \{1, 2\}$, yielding negligible runtime per patient.

4 Main Results

4.1 Theoretical Foundations and Stability Analysis

This section presents theorems, that validate the proposed framework. The theorems are established for the general case of $p \geq 1$ acoustic parameters, each having $q_r \geq 2$ linguistic categories. This demonstrates that the results apply to arbitrary dimensions and are not restricted to the specific $3 \times 3 = 9$ rule structure used in the numerical study.

Theorem 4.1. Let $p \geq 1$ acoustic parameters be given. For each parameter $r \in \{1, \ldots, p\}$, let there be $q_r \geq 2$ linguistic categories with membership functions $\mu_{r,\ell} : U \to [0, 1]$ for $\ell \in \{1, \ldots, q_r\}$. Form $n = \prod_{r=1}^{p} q_r$ fuzzy soft rules via the p-ary minimum:

$$R_j(x_i) = \min\bigl\{\mu_{1,\ell_{j,1}}(x_i),\, \mu_{2,\ell_{j,2}}(x_i),\, \ldots,\, \mu_{p,\ell_{j,p}}(x_i)\bigr\}, \tag{9}$$

where $(\ell_{j,1}, \ldots, \ell_{j,p})$ enumerates all n combinations of linguistic categories.

Let $\mathcal{K}(x_i) = \{\, j : R_j(x_i) > 0 \,\}$ be the set of active rules and let $k = |\mathcal{K}(x_i)| \geq 1$. The aggregated fuzzy soft rule interaction is defined by:

$$\text{Agg}(x_i) = \frac{1}{k} \sum_{j \in \mathcal{K}(x_i)} R_j(x_i). \tag{10}$$

Then for every patient x_i, all $p \geq 1$, and all $q_r \geq 2$:

$$0 \leq \mathrm{Agg}(x_i) \leq \max_{j \in \mathcal{K}(x_i)} R_j(x_i) \leq 1. \tag{11}$$

Proof. For each rule j, since $0 \leq \mu_{r,\ell}(x_i) \leq 1$ for all r and ℓ, the p-ary minimum satisfies:

$$0 \leq R_j(x_i) = \min\big\{\mu_{1,\ell_{j,1}}(x_i), \ldots, \mu_{p,\ell_{j,p}}(x_i)\big\} \leq 1. \tag{12}$$

The lower bound follows because the minimum of non-negative numbers is non-negative. The upper bound follows because the minimum of numbers each bounded above by 1 is itself bounded above by 1.

For all $j \in \mathcal{K}(x_i)$, we have $0 < R_j(x_i) \leq 1$ by (12). Summing over all k active rules:

$$0 < \sum_{j \in \mathcal{K}(x_i)} R_j(x_i) \leq \sum_{j \in \mathcal{K}(x_i)} 1 = k.$$

Dividing by $k \geq 1$ as prescribed by Equation (10):

$$0 < \frac{1}{k} \sum_{j \in \mathcal{K}(x_i)} R_j(x_i) \leq 1,$$

which yields $0 < \mathrm{Agg}(x_i) \leq 1$. The weak inequality $0 \leq \mathrm{Agg}(x_i) \leq 1$ follows immediately.

Let $M(x_i) = \max_{j \in \mathcal{K}(x_i)} R_j(x_i)$. By definition of the maximum, $R_j(x_i) \leq M(x_i)$ for every $j \in \mathcal{K}(x_i)$. Summing over all k active rules:

$$\sum_{j \in \mathcal{K}(x_i)} R_j(x_i) \leq \sum_{j \in \mathcal{K}(x_i)} M(x_i) = k \cdot M(x_i).$$

Dividing both sides by $k > 0$:

$$\mathrm{Agg}(x_i) = \frac{1}{k} \sum_{j \in \mathcal{K}(x_i)} R_j(x_i) \leq M(x_i) = \max_{j \in \mathcal{K}(x_i)} R_j(x_i). \qquad \square$$

Remark 4.2. The proof uses only that: (i) each $R_j \in [0, 1]$, (ii) $k \geq 1$, and (iii) the arithmetic mean is used. None of these depend on $p = 2$ or $q_r = 3$. The theorem holds for any $p \geq 1$ parameters with any $q_r \geq 2$ categories each, yielding any $n = \prod q_r \geq 4$ rules.

Theorem 4.3. Let the setup of Theorem 4.1 hold with $p \geq 1$ parameters and $n = \prod_{r=1}^{p} q_r$ rules defined by Equation (9). Consider perturbed membership values $\mu'_{r,\ell}(x_i) \geq \mu_{r,\ell}(x_i)$ for all $r \in \{1, \ldots, p\}$ and $\ell \in \{1, \ldots, q_r\}$. Let primed quantities denote the corresponding outputs. Then:

(i) If $\mathcal{A}' = \mathcal{A}$ (the set of active rules is unchanged), then $\mathrm{Agg}'(x_i) \geq \mathrm{Agg}(x_i)$.

(ii) Under the hypothesis that all membership values are non-decreasing, $k' \geq k$ (the number of active rules cannot decrease).

(iii) If $\mathcal{A}' \supseteq \mathcal{A}$ and all newly active rules satisfy $R'_j(x_i) \geq \mathrm{Agg}(x_i)$, then $\mathrm{Agg}'(x_i) \geq \mathrm{Agg}(x_i)$.

Proof. The p-ary minimum operator is non-decreasing in each argument. By induction on p from the binary case: for any $a'_r \geq a_r$ for all $r \in \{1, \ldots, p\}$,

$$\min\{a'_1, \ldots, a'_p\} \geq \min\{a_1, \ldots, a_p\}.$$

Applying this to the rule strength definitions with $\mu'_{r,\ell_{j,r}} \geq \mu_{r,\ell_{j,r}}$:

$$R'_j(x_i) = \min\bigl\{\mu'_{1,\ell_{j,1}}, \ \ldots, \ \mu'_{p,\ell_{j,p}}\bigr\} \geq \min\bigl\{\mu_{1,\ell_{j,1}}, \ \ldots, \ \mu_{p,\ell_{j,p}}\bigr\} = R_j(x_i). \tag{13}$$

Thus $R'_j(x_i) \geq R_j(x_i)$ for every rule $j \in \{1, \ldots, n\}$.

Proof of (i). If $\mathcal{A}' = \mathcal{A}$, then $k' = k$ and:

$$\mathrm{Agg}'(x_i) = \frac{1}{k} \sum_{j \in \mathcal{A}} R'_j(x_i) \geq \frac{1}{k} \sum_{j \in \mathcal{A}} R_j(x_i) = \mathrm{Agg}(x_i),$$

where the inequality follows from (13).

Proof of (ii). For any $j \in \mathcal{A}$, we have $R_j(x_i) > 0$. By (13), $R'_j(x_i) \geq R_j(x_i) > 0$, so $j \in \mathcal{A}'$. Therefore $\mathcal{A} \subseteq \mathcal{A}'$, which implies $k' = |\mathcal{A}'| \geq |\mathcal{A}| = k$.

Proof of (iii). Decompose $\mathcal{A}' = \mathcal{A} \cup (\mathcal{A}' \setminus \mathcal{A})$ (disjoint union). Then:

$$\mathrm{Agg}'(x_i) = \frac{1}{k'} \left[\sum_{j \in \mathcal{A}} R'_j(x_i) + \sum_{j \in \mathcal{A}' \setminus \mathcal{A}} R'_j(x_i) \right].$$

Since $R'_j(x_i) \geq R_j(x_i)$ for $j \in \mathcal{A}$: $\sum_{j \in \mathcal{A}} R'_j(x_i) \geq k \cdot \mathrm{Agg}(x_i)$. If each newly active rule satisfies $R'_j(x_i) \geq \mathrm{Agg}(x_i)$, then with $m = |\mathcal{A}' \setminus \mathcal{A}| = k' - k$:

$$\mathrm{Agg}'(x_i) \geq \frac{k \cdot \mathrm{Agg}(x_i) + m \cdot \mathrm{Agg}(x_i)}{k + m} = \mathrm{Agg}(x_i). \qquad \square$$

Theorem 4.4. Let $p \geq 2$ acoustic parameters be given. For each parameter $r \in \{1, \ldots, p\}$, let:

$$D_r(x_i) = \max_{\ell \in \{1, \ldots, q_r\}} \mu_{r,\ell}(x_i) \in [0, 1] \tag{14}$$

be the dominant membership for parameter r. Define the generalized RMS instability:

$$\mathrm{Ins}(x_i) = \sqrt{\frac{1}{p} \sum_{r=1}^{p} \bigl(D_r(x_i)\bigr)^2}. \tag{15}$$

Then:

$$0 \leq \mathrm{Ins}(x_i) \leq 1, \tag{16}$$

with $\mathrm{Ins}(x_i) = 0 \Leftrightarrow D_r(x_i) = 0$ for all r, and $\mathrm{Ins}(x_i) = 1 \Leftrightarrow D_r(x_i) = 1$ for all r.

Proof. Since each $\mu_{r,\ell}(x_i) \in [0,1]$, each $D_r(x_i) \in [0,1]$.

From $D_r(x_i) \geq 0$, we have $(D_r)^2 \geq 0$ for all r. Hence $\frac{1}{p}\sum_{r=1}^{p}(D_r)^2 \geq 0$, and since $\sqrt{\cdot}$ is monotone increasing with $\sqrt{0} = 0$:

$$\mathrm{Ins}(x_i) \geq 0.$$

From $D_r(x_i) \leq 1$, squaring gives $(D_r)^2 \leq 1$. Summing over p parameters:

$$\sum_{r=1}^{p}(D_r)^2 \leq \sum_{r=1}^{p} 1 = p \quad \Rightarrow \quad \frac{1}{p}\sum_{r=1}^{p}(D_r)^2 \leq 1.$$

Applying the monotone square root: $\mathrm{Ins}(x_i) \leq \sqrt{1} = 1$.

($\Rightarrow$) If $\mathrm{Ins}(x_i) = 0$, then $\sum(D_r)^2 = 0$. Since each $(D_r)^2 \geq 0$, each must be zero: $D_r(x_i) = 0$ for all r. ($\Leftarrow$) If all $D_r = 0$, then $\mathrm{Ins}(x_i) = \sqrt{\frac{1}{p} \cdot 0} = 0$.

($\Rightarrow$) If $\mathrm{Ins}(x_i) = 1$, then $\sum(D_r)^2 = p$. Since each $(D_r)^2 \leq 1$, equality requires $(D_r)^2 = 1$ for all r, so $D_r(x_i) = 1$ for all r. ($\Leftarrow$) If all $D_r = 1$, then $\mathrm{Ins}(x_i) = \sqrt{\frac{1}{p} \cdot p} = 1$. $\square$

Corollary 4.5. For $p = 2$ with $D_1 = \mathrm{FFD}$ and $D_2 = \mathrm{VPID}$:

$$\mathrm{Ins}(x_i) = \sqrt{\frac{1}{2}\big[(\mathrm{FFD})^2 + (\mathrm{VPID})^2\big]} = \sqrt{0.5\big[(\mathrm{FFD})^2 + (\mathrm{VPID})^2\big]},$$

which is exactly Equation (5).

Theorem 4.6. Let $Q = \sum_{r=1}^{p} q_r$ be the total number of membership values. Define the vector $\mathbf{m} = (\mu_{1,1}, \ldots, \mu_{p,q_p}) \in [0,1]^Q$. For each parameter r, define $D_r(\mathbf{m}) = \max_\ell \mu_{r,\ell}$. Define:

$$\mathcal{I}(\mathbf{m}) = \sqrt{\frac{1}{p}\sum_{r=1}^{p}\big(D_r(\mathbf{m})\big)^2}.$$

Then $\mathcal{I} : [0,1]^Q \to [0,1]$ is continuous on its entire domain for all $p \geq 2$ and all $q_r \geq 2$.

Proof. Continuity of projections. Each coordinate projection $\pi_j(\mathbf{m}) = \mu_j$ satisfies $|\pi_j(\mathbf{m}) - \pi_j(\mathbf{m}')| = |\mu_j - \mu_j'| \leq d_Q(\mathbf{m}, \mathbf{m}')$, hence is Lipschitz (therefore continuous).

Continuity of D_r. Each D_r is a q_r-ary maximum of continuous functions. The binary $\max\{a,b\} = \frac{1}{2}(a + b + |a - b|)$ is continuous. The q_r-ary maximum is a composition of $q_r - 1$ binary max operations, hence continuous. Moreover, D_r is 1-Lipschitz:

$$|D_r(\mathbf{m}) - D_r(\mathbf{m}')| \leq \max_\ell |\mu_{r,\ell} - \mu_{r,\ell}'| \leq d_Q(\mathbf{m}, \mathbf{m}')$$

.

The map $(D_1, \ldots, D_p) \mapsto \sqrt{\frac{1}{p}\sum D_r^2}$ is a composition of: (i) squaring (polynomial, continuous), (ii) averaging (linear, continuous), (iii) square root (continuous on $[0, \infty)$). Since $D_r \in [0, 1]$, the argument of $\sqrt{\cdot}$ lies in $[0, 1] \subset [0, \infty)$.

$\mathcal{I}$ is the composition of the continuous map $\mathbf{m} \mapsto (D_1, \ldots, D_p)$ and the continuous RMS function. The composition of continuous functions is continuous. $\square$

Remark 4.7. The continuity of $\mathcal{I}$ guarantees that small measurement errors in acoustic parameters produce only small changes in the instability score. Formally: for any precision $\varepsilon > 0$, there exists $\delta > 0$ such that if all membership values are measured within δ of their true values, the instability score is within ε of the true score. This ensures robustness against instrument noise and inter-rater variability.

Theorem 4.8. Let $p \geq 1$, $q_1, \ldots, q_p \geq 2$, $n = \prod q_r$, and $Q = \sum q_r$. Let all Q membership values be perturbed by at most $\varepsilon > 0$: $|\mu'_{r,\ell}(x_i) - \mu_{r,\ell}(x_i)| \leq \varepsilon$ for all r, ℓ. Let Agg and Ins be defined by Equations (10) and (15). Then:

$$\big|\operatorname{Agg}'(x_i) - \operatorname{Agg}(x_i)\big| \leq \varepsilon, \qquad \big|\operatorname{Ins}'(x_i) - \operatorname{Ins}(x_i)\big| \leq \varepsilon. \tag{17}$$

The combined system $(\operatorname{Agg}, \operatorname{Ins})$ is jointly perturbation stable with Lipschitz constant $L = 1$.

Proof.

Lemma 4.9. For $p \geq 2$ and $a_1, \ldots, a_p, a'_1, \ldots, a'_p \in \mathbb{R}$ with $|a'_r - a_r| \leq \varepsilon$:

$$\big|\min\{a'_1, \ldots, a'_p\} - \min\{a_1, \ldots, a_p\}\big| \leq \varepsilon.$$

Proof. By induction on p. Base case $p = 2$: four-case analysis identical to Theorem 5, Lemma 1. Inductive step: write $\min\{a_1, \ldots, a_p\} = \min\{\min\{a_1, \ldots, a_{p-1}\}, a_p\}$. By hypothesis, $|\min\{a'_1, \ldots, a'_{p-1}\} - \min\{a_1, \ldots, a_{p-1}\}| \leq \varepsilon$. Apply the $p = 2$ case. $\square$

Lemma 4.10. For $q \geq 1$ with $\max_\ell |a'_\ell - a_\ell| \leq \varepsilon$: $\big|\max_\ell a'_\ell - \max_\ell a_\ell\big| \leq \varepsilon$.

Proof. Let $M = \max_\ell a_\ell$, $M' = \max_\ell a'_\ell$. WLOG $M' \geq M$. Let j satisfy $a'_j = M'$. Then $M' - M = a'_j - M = (a'_j - a_j) + (a_j - M) \leq |a'_j - a_j| + 0 \leq \varepsilon$. $\square$

Lemma 4.11. For $f(u_1, \ldots, u_p) = \sqrt{\frac{1}{p}\sum u_r^2} = \frac{1}{\sqrt{p}}\|\mathbf{u}\|_2$:

$$|f(\mathbf{u}') - f(\mathbf{u})| \leq \frac{1}{\sqrt{p}}\|\mathbf{u}' - \mathbf{u}\|_2.$$

Proof. By the reverse triangle inequality for $\|\cdot\|_2$ on $\mathbb{R}^p$. $\square$

By Lemma 4.9, $|R'_j - R_j| \le \varepsilon$ for each rule. Then:

$$|\,\mathrm{Agg}' - \mathrm{Agg}\,| = \left|\frac{1}{k}\sum_{j=1}^{k}(R'_j - R_j)\right| \le \frac{1}{k}\sum_{j=1}^{k}|R'_j - R_j| \le \frac{k\varepsilon}{k} = \varepsilon.$$

By Lemma 4.10, $|D'_r - D_r| \le \varepsilon$ for each r. By Lemma 4.11:

$$|\,\mathrm{Ins}' - \mathrm{Ins}\,| \le \frac{1}{\sqrt{p}}\sqrt{\sum_{r=1}^{p}(D'_r - D_r)^2} \le \frac{1}{\sqrt{p}}\sqrt{\sum_{r=1}^{p}\varepsilon^2} = \frac{1}{\sqrt{p}}\sqrt{p\varepsilon^2} = \varepsilon.$$

$\|\Phi' - \Phi\|_\infty = \max\{|\,\mathrm{Agg}' - \mathrm{Agg}\,|, |\,\mathrm{Ins}' - \mathrm{Ins}\,|\} \le \varepsilon.$ □

Remark 4.12. The constants $C_1 = 1$ and $C_2 = 1$ are **independent** of p, q_r, n, and Q. The Lipschitz constant $L = 1$ is **universal** — it does not grow with dimension. Even with 10 parameters and 100 membership values, a measurement error of ε cannot produce an output error larger than ε.

Note Crucially, all theorems are proved for arbitrary dimensions p, q_r, and n, not merely the $p = 2$, $q_1 = q_2 = 3$, $n = 9$ configuration employed in the numerical study. This means the theoretical guarantees remain valid when additional acoustic parameters (jitter, shimmer, HNR, NHR) or finer linguistic taxonomies are incorporated into the framework. The bounds, monotonicity, continuity, and stability properties are intrinsic to the mathematical structure of aggregated fuzzy soft rule interaction and RMS instability modelling, not artifacts of the particular 3×3 parameterization chosen for illustration.

4.2 Numerical Computationals

This section illustrates the complete computational workflow for patients x_1 and x_{10}, adapted from Okigbo et al. (2026).

Patient x_1

Given fuzzification outputs:

$$\begin{array}{lll} \mu_{\mathrm{FF,AT}}(x_1) = 0 & \mu_{\mathrm{FF,NT}}(x_1) = 0 & \mu_{\mathrm{FF,ST}}(x_1) = 0.341 \\ \mu_{\mathrm{VPI,NV}}(x_1) = 0.659 & \mu_{\mathrm{VPI,RV}}(x_1) = 0 & \mu_{\mathrm{VPI,AV}}(x_1) = 0 \end{array}$$

Dominant membership extraction:

$$\mathrm{FFD}(x_1) = \max(0,\, 0,\, 0.341) = 0.341,$$
$$\mathrm{VPID}(x_1) = \max(0.659,\, 0,\, 0) = 0.659.$$

Hence $(\mathrm{FFD}(x_1), \mathrm{VPID}(x_1)) = (0.341,\, 0.659)$.

Rule evaluation: Active memberships are ST = 0.341 and NV = 0.659. Thus:

$$R_7(x_1) = \mathrm{FF}_{\mathrm{ST}}(x_1) \wedge \mathrm{VPI}_{\mathrm{NV}}(x_1) = \min(0.341,\, 0.659) = 0.341.$$

Since only one rule is active, $k = 1$.

Aggregated rule computation:

$$\mathrm{Agg}(x_1) = \frac{0.341}{1} = 0.341.$$

RMS acoustic instability computation:

$$\begin{aligned}\mathrm{Ins}(x_1) &= \sqrt{0.5 \cdot (0.341^2 + 0.659^2)} = \sqrt{0.5 \cdot (0.1163 + 0.4343)} \\ &= \sqrt{0.5 \cdot 0.5506} = \sqrt{0.2753} \approx \mathbf{0.525}.\end{aligned}$$

Patient x_{10}

Given fuzzification outputs:

$$\begin{array}{lll} \mu_{\mathrm{FF,AT}}(x_{10}) = 0 & \mu_{\mathrm{FF,NT}}(x_{10}) = 0.729 & \mu_{\mathrm{FF,ST}}(x_{10}) = 0 \\ \mu_{\mathrm{VPI,NV}}(x_{10}) = 0 & \mu_{\mathrm{VPI,RV}}(x_{10}) = 0.055 & \mu_{\mathrm{VPI,AV}}(x_{10}) = 0.201 \end{array}$$

Dominant membership extraction:

$$\begin{aligned}\mathrm{FFD}(x_{10}) &= \max(0,\, 0.729,\, 0) = 0.729, \\ \mathrm{VPID}(x_{10}) &= \max(0,\, 0.055,\, 0.201) = 0.201.\end{aligned}$$

Hence $(\mathrm{FFD}(x_{10}), \mathrm{VPID}(x_{10})) = (0.729,\, 0.201)$.

Rule evaluation: Active memberships are NT = 0.729, RV = 0.055, AV = 0.201. Thus:

$$\begin{aligned}R_5(x_{10}) &= \min(0.729,\, 0.055) = 0.055, \\ R_6(x_{10}) &= \min(0.729,\, 0.201) = 0.201.\end{aligned}$$

Since two rules are active, $k = 2$.

Aggregated rule computation:

$$\mathrm{Agg}(x_{10}) = \frac{0.055 + 0.201}{2} = \frac{0.256}{2} = \mathbf{0.128}.$$

RMS acoustic instability computation:

$$\begin{aligned}\mathrm{Ins}(x_{10}) &= \sqrt{0.5 \cdot (0.729^2 + 0.201^2)} = \sqrt{0.5 \cdot (0.5314 + 0.0404)} \\ &= \sqrt{0.5 \cdot 0.5718} = \sqrt{0.2859} \approx \mathbf{0.535}.\end{aligned}$$

Table 2: Aggregated fuzzy soft rule interaction and RMS instability values for all 20 patients

Patient	FFD(x_i)	VPID(x_i)	Agg(x_i)	Ins(x_i)	$k(x_i)$
x_1	0.341	0.659	0.341	0.525	1
x_2	0.400	0.790	0.400	0.626	1
x_3	0.729	0.677	0.677	0.703	1
x_4	0.280	0.830	0.280	0.619	1
x_5	0.126	0.571	0.126	0.413	1
x_6	0.706	0.527	0.527	0.623	1
x_7	0.250	0.430	0.250	0.352	1
x_8	0.526	0.943	0.526	0.764	1
x_9	0.329	0.940	0.329	0.704	1
x_{10}	0.729	0.201	0.128	0.535	2
x_{11}	0.118	0.427	0.085	0.313	2
x_{12}	0.920	0.500	0.265	0.740	2
x_{13}	0.824	0.866	0.824	0.845	1
x_{14}	0.447	0.742	0.447	0.613	1
x_{15}	0.471	0.571	0.471	0.523	1
x_{16}	0.400	0.866	0.400	0.675	1
x_{17}	0.870	0.890	0.870	0.880	1
x_{18}	0.824	0.211	0.175	0.601	2
x_{19}	0.965	0.988	0.965	0.977	1
x_{20}	0.432	0.232	0.181	0.347	2

4.3 Statistical Validation

This section provides rigorous statistical analysis of the computational results presented in Table 2, quantifying the relationships between aggregation, instability, and the underlying dominant memberships.

4.3.1 Pearson Correlation Analysis

Pearson correlation coefficients were computed to assess linear relationships between key variables. The null hypothesis $H_0 : \rho = 0$ was tested against the two-sided alternative $H_1 : \rho \neq 0$ at significance level $\alpha = 0.05$.

Table 3: Pearson correlation coefficients ($n = 20$ patients)

Variable Pair	r	p**-value**	**Interpretation**
Agg vs. Ins	0.8267	< 0.0001	Strong positive
FFD vs. Ins	0.7333	0.0002	Strong positive
VPID vs. Ins	0.7050	0.0005	Strong positive

The correlation between aggregation and instability ($r = 0.827$, $p < 0.001$) is both statistically significant and substantively meaningful. This confirms that patients with stronger rule-based parameter coupling also exhibit greater acoustic oscillatory variability, validating the dual-measure design of the framework. The high correlation does *not* imply redundancy: Agg measures logical rule interaction strength while Ins quantifies Euclidean energy of oscillatory variability they capture complementary aspects of a shared underlying vocal vulnerability construct. The correlations between each dominant membership and instability (FFD: $r = 0.733$, $p < 0.001$; VPID: $r = 0.705$, $p < 0.001$) confirm that both acoustic parameters contribute meaningfully to the instability score. The slightly higher correlation for FFD suggests that fundamental frequency variation may be the stronger driver of acoustic instability in this dataset, though both parameters remain essential.

4.3.2 Descriptive Statistics

Table 4 presents descriptive statistics for all four primary variables across the 20-patient cohort.

Table 4: Descriptive statistics for $n = 20$ patients

Variable	**Mean**	**SD**	**Min**	**Max**
$\text{Agg}(x_i)$	0.4133	0.2558	0.085	0.965
$\text{Ins}(x_i)$	0.6189	0.1788	0.313	0.977
$\text{FFD}(x_i)$	0.5343	0.2659	0.118	0.965
$\text{VPID}(x_i)$	0.6430	0.2526	0.201	0.988

The mean instability (0.619 ± 0.179) exceeds the mean aggregation (0.413 ± 0.256), reflecting the fact that Ins incorporates both FFD and VPID through the RMS energy formulation, while Agg averages only the active rule strengths. The coefficient of variation (CV) is higher for aggregation (CV = 61.9%) than for instability (CV = 28.9%), indicating that Agg discriminates more effectively between patients across the full dynamic range. The near-maximum values for FFD and VPID (0.965 and 0.988 respectively) confirm that the dataset spans nearly the entire theoretical range $[0, 1]$, providing a comprehensive test of the framework's bounds.

Remark 4.13. The strong positive correlations and wide dynamic range together confirm that the proposed framework produces diagnostically informative scores with good discriminatory power. The statistical significance of all correlations ($p < 0.001$) exceeds the conventional threshold by a substantial margin, even with the modest sample size of $n = 20$.

Table 5: Sensitivity analysis: empirical perturbation results for 20 patients ($n = 40{,}000$ trials)

ε	$\max\lvert\Delta\,\text{Agg}\rvert$	$\max\lvert\Delta\,\text{Ins}\rvert$	L_{Agg}	L_{Ins}
0.01	0.0100	0.0099	0.4904	0.3408
0.05	0.0500	0.0499	0.4782	0.3411
0.10	0.1000	0.0983	0.4812	0.3337

The results in Table 5 confirm the theoretical guarantee of Theorem 4.8: the maximum observed change in both Agg and Ins never exceeds the perturbation bound ε for any of the three tested magnitudes. The empirical Lipschitz constants are substantially below the theoretical worst-case bound $L = 1$, indicating that typical perturbation effects are far smaller than the worst-case scenario.

Remark 4.14. The empirical Lipschitz constants $L_{\text{Agg}} \approx 0.48$ and $L_{\text{Ins}} \approx 0.34$ reveal that, in practice, the system is substantially more robust than the theoretical bound suggests. The RMS instability measure is particularly resilient: on average, a membership perturbation of magnitude ε produces an instability change of only approximately 0.34ε. This practical robustness arises because the RMS formula (Eq. (15)) incorporates the $\frac{1}{\sqrt{p}}$-Lipschitz property of the Euclidean norm, which dampens perturbation propagation.

Corollary 4.15. With measurement instrumentation typically producing errors $\varepsilon \leq 0.05$ (5% of full scale), the maximum expected diagnostic score change is bounded by 0.05. For the RMS instability, the *typical* change is approximately $0.34 \times 0.05 = 0.017$, well within the tolerance of clinical decision-making. This confirms the framework's suitability for real-world deployment with standard acoustic measurement equipment.

5 Discussion

The computational results in Table 2 reveal three important patterns. First, patients with high dominant memberships in both FF and VPI categories (e.g., FFD > 0.9 and VPID > 0.9) consistently produce high aggregation and RMS instability values, indicating strong vocal vulnerability. These cases approach the theoretical upper bound Ins $= 1$ predicted by Theorem 4.4. Conversely, patients with low dominant memberships in both parameters yield low instability scores near the lower bound, consistent with acoustically stable vocal function. Second, patients with a single active rule ($k = 1$) show Agg $= \max_j R_j$, confirming the sharpness of the upper bound in Theorem 4.1. Patients with multiple active rules ($k \geq 2$) demonstrate how aggregation dampens individual rule strengths while preserving their collective contribution. This averaging behavior addresses the core limitation of classical maximum-rule systems, which discard all but the strongest activation. Third, the RMS instability values consistently exceed or differ from the aggregated interaction values across all patients. This confirms that aggregation and RMS capture complementary dimensions of vocal vulnerability: aggregation measures logical rule-based interaction strength, while RMS quantifies the Euclidean energy of oscillatory variability. Neither measure alone provides a complete diagnostic picture; their combined use offers a more nuanced assessment framework.

5.1 Future Work

The proposed framework opens several avenues for future investigation.

Real-time clinical validation. Prospective studies with blinded laryngologist assessments are needed to validate the diagnostic categories against clinical ground truth. Time-series validation, where patients are monitored longitudinally, would also assess whether changes in Agg and Ins track the progression or remission of vocal disorders.

Extension to other biomedical signals. The aggregated fuzzy soft RMS framework is not specific to vocal acoustics. The same mathematical structure applies to EEG signal analysis (where p electrodes serve as parameters), ECG arrhythmia detection (where p morphological features are monitored), and respiratory sound analysis. Cross-domain adaptation requires only domain-specific fuzzification and threshold recalibration.

Mobile health (mHealth) deployment. The $O(n)$ computational complexity of Algorithm 1 and the bounded output range $[0, 1]$ make the framework well-suited for mobile deployment. A smartphone application could record vocal samples, compute fuzzification outputs using lightweight models, and execute the complete aggregated fuzzy soft RMS pipeline in real time, enabling remote vocal monitoring for at-risk populations.

Cross-population studies. Age-specific, gender-specific, and profession-specific models (e.g., singers, teachers, call-center operators) may exhibit different baseline distributions of Agg and Ins. Cross-population studies with stratified sampling are needed to establish population-normative ranges and tailor the diagnostic thresholds accordingly.

6 Conclusion

This paper presented an aggregated fuzzy soft rule interaction and RMS based acoustic instability framework for vocal vulnerability assessment. The proposed aggregation operator averages all active fuzzy rules rather than selecting only the dominant rule, thereby preserving overlapping acoustic interactions that conventional systems discard. An RMS formulation quantifies oscillatory variability through dominant memberships extracted from acoustic parameters. A complete algorithmic specification (Algorithm 1) was provided, along with diagnostic classification thresholds that translate continuous scores into clinically actionable categories. Statistical validation on 20 patients confirmed strong positive correlations between aggregation and instability ($r = 0.827$, $p < 0.001$), while sensitivity analysis verified the theoretical perturbation bound $L = 1$ and revealed superior practical robustness with empirical Lipschitz constants of approximately 0.48 for aggregation and 0.34 for instability.

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