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AVERAGE AND EXTREMAL VALUES OF COMPLETE WORK-DONE ON TRANSFORMATION SEMIGROUPS

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Abstract. In this paper, we study a complete work-done function defined on three important transformation semigroups: the full transformation semigroup T_n, the partial transformation semigroup P_n, and the symmetric inverse semigroup I_n. For each class, closed formulae are obtained for the total and average values of the complete work-done function. Extremal values are also determined for T_n and P_n. The results show that algebraic restrictions such as partiality and injectivity significantly influence the magnitude and distribution of the complete work-done values. Numerical tables and graphical illustrations are provided for $1 \leq n \leq 10$ to support the theoretical findings.

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1. Introduction and preliminaries

Transformation semigroups form an important class of algebraic structures in finite semigroup theory and algebraic combinatorics. A transformation on a finite set is a mapping from the set into itself, and the collection of all such mappings under composition forms a semigroup. These semigroups provide a natural framework for studying mappings, ranks, kernels, images, idempotents, Green's relations, and related combinatorial properties. The general theory of semigroups and transformation semigroups has been extensively developed in the literature [7, 9].

Let

$$[n] = \{1, 2, \ldots, n\}.$$

The full transformation semigroup on $[n]$, denoted by T_n, consists of all mappings from $[n]$ into itself. Thus,

$$T_n = \{\alpha : \alpha : [n] \to [n]\},$$

and its cardinality is

$$|T_n| = n^n.$$

If transformations are allowed to be undefined on some elements of the domain, one obtains the partial transformation semigroup P_n. Thus, each $\alpha \in P_n$ is a mapping

$$\alpha : \mathrm{dom}(\alpha) \to [n],$$

where $\mathrm{dom}(\alpha) \subseteq [n]$. Since each element of $[n]$ may either be undefined or mapped to one of the n elements of $[n]$, we have

$$|P_n| = (n+1)^n.$$

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The symmetric inverse semigroup I_n consists of all partial one-to-one transformations on $[n]$. Its cardinality is

$$|I_n| = \sum_{r=0}^{n} \binom{n}{r}^2 r!,$$

since a partial one-to-one transformation of rank r is obtained by choosing r domain elements, r image elements, and then a bijection between them [12,14–16].

Several authors have investigated the algebraic and combinatorial properties of transformation semigroups and their important subsemigroups. Gomes and Howie studied the ranks of certain semigroups of order-preserving transformations [8]. Laradji and Umar obtained combinatorial results for semigroups of order-preserving partial transformations and order-decreasing partial transformations [10, 11]. Adeniji and Makanjuola considered collapse and height properties in full transformation semigroups [1]. Adeshola investigated full contraction mappings of a finite chain [2], while Mogbonju, Adeniji and Ogunleke studied signed singular mapping transformation semigroups [13]. These works provide useful background for the enumeration and structural analysis of finite transformation semigroups. In addition to structural properties, quantitative measures have also become useful in the study of transformation semigroups. Such measures describe not only the algebraic type of a transformation but also the amount of displacement caused by the transformation. One important example is the work performed by a transformation semigroup. East and McNamara introduced and studied the work performed by transformation semigroups and obtained total and average work formulae for several classes of transformation semigroups [3]. This approach gives a numerical interpretation of how elements are moved under transformations. Further developments of this idea have appeared in more recent works. Francis, Adeniji and Mogbonju studied workdone by m-topological transformation semigroup regular spaces ($M_{\psi n}$) [4]. Francis and Adeniji investigated work performed by closed and clopen m-topological full transformation semigroup spaces MCT_n and $Clp(MCT_n)$ [5]. More recently, Francis and Ranaivomanana introduced the concept of cross work performed by partial transformations [6]. Their work extended the displacement idea by measuring interactions between neighbouring domain-image positions.

For a transformation α, ordinary work-done is based on the pointwise displacement between an element and its image. For $\alpha \in T_n$, the pointwise work-done is given by

$$w(\alpha) = \sum_{i=1}^{n} |i - \alpha(i)|.$$

For partial transformations, if $i \notin \mathrm{dom}(\alpha)$, the contribution of i to the pointwise work may be taken to be zero. Thus, for $\alpha \in P_n$, define

$$d_\alpha(i) = \begin{cases} |i - \alpha(i)|, & i \in \mathrm{dom}(\alpha), \\ 0, & i \notin \mathrm{dom}(\alpha), \end{cases}$$

and

$$w(\alpha) = \sum_{i=1}^{n} d_\alpha(i).$$

For a semigroup $S \subseteq P_n$, the total and average pointwise work-done are denoted by

$$W(S) = \sum_{\alpha \in S} w(\alpha), \qquad \overline{w}(S) = \frac{W(S)}{|S|}.$$

The cross work-done function introduced in [6] measures displacement across adjacent positions. For $\alpha \in P_n$, it is convenient to use the zero extension

$$\widehat{\alpha} : [n] \to \{0, 1, \ldots, n\}$$

defined by

$$\widehat{\alpha}(j) = \begin{cases} \alpha(j), & j \in \mathrm{dom}(\alpha), \\ 0, & j \notin \mathrm{dom}(\alpha). \end{cases}$$

Then, for $i = 1, 2, \ldots, n-1$, the left and right cross distances are

$$\delta_i^L(\alpha) = |i - \widehat{\alpha}(i+1)|, \qquad \delta_i^R(\alpha) = |(i+1) - \widehat{\alpha}(i)|.$$

The local cross work is

$$w_i^{\times}(\alpha) = \delta_i^L(\alpha) + \delta_i^R(\alpha),$$

that is,

$$w_i^{\times}(\alpha) = |i - \widehat{\alpha}(i+1)| + |(i+1) - \widehat{\alpha}(i)|.$$

The total cross work-done is

$$w^{\times}(\alpha) = \sum_{i=1}^{n-1} w_i^{\times}(\alpha).$$

The present paper continues this line of investigation by introducing and studying the complete work-done function on T_n, P_n, and I_n. Unlike pointwise work-done, which compares i only with $\alpha(i)$, the complete work-done compares every element $i \in [n]$ with every image value of the transformation. Thus, for $\alpha \in P_n$, define the local complete work at i by

$$w_i^{\varphi}(\alpha) = \sum_{j=1}^{n} \bigl|i - \widehat{\alpha}(j)\bigr|.$$

The complete work-done of α is

$$w^{\varphi}(\alpha) = \sum_{i=1}^{n} w_i^{\varphi}(\alpha).$$

Equivalently,

$$w^{\varphi}(\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} \bigl|i - \widehat{\alpha}(j)\bigr|.$$

If $\alpha \in T_n$, then $\widehat{\alpha} = \alpha$, and so

$$w^{\varphi}(\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} |i - \alpha(j)|.$$

For any semigroup $S \in \{T_n, P_n, I_n\}$, the total complete work-done and the average complete work-done are denoted respectively by

$$W^{\varphi}(S) = \sum_{\alpha \in S} w^{\varphi}(\alpha), \qquad \overline{W^{\varphi}}(S) = \frac{W^{\varphi}(S)}{|S|}.$$

The motivation for studying w^{φ} is that it gives a global measure of displacement. It does not only measure how far each point is moved by its own image; rather, it compares every point in the finite set with every image value of the transformation. Therefore, complete work-done captures more positional information than ordinary pointwise work-done and cross work-done. It also provides a common quantitative framework for comparing the full transformation semigroup T_n, the partial transformation semigroup P_n, and the symmetric inverse semigroup I_n.

The main objective of this paper is to derive closed formulae for the total and average values of w^{φ} over T_n, P_n, and I_n. We also determine extremal values of

w^{φ} on T_n and P_n. The results show that partiality and injectivity influence the size and distribution of complete work-done values. In particular, undefined elements in partial transformations contribute additional displacement through the value 0, while injectivity affects the distribution of image values.

We now record a useful distance function that will be used in the proofs.

Definition 1.1. *For $v \in \{0, 1, \ldots, n\}$, define*

$$D_n(v) = \sum_{i=1}^{n} |i - v|.$$

In particular, if $v \in [n]$, then

$$D_n(v) = \frac{(v-1)v}{2} + \frac{(n-v)(n-v+1)}{2}.$$

Also,

$$D_n(0) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Lemma 1.2. *The following identity holds:*

$$\sum_{i=1}^{n}\sum_{k=1}^{n} |i-k| = \frac{n(n^2-1)}{3}.$$

Proof. We have

$$\sum_{i=1}^{n}\sum_{k=1}^{n} |i-k| = 2 \sum_{1 \le i < k \le n} (k-i).$$

Let $d = k - i$. For each $d = 1, 2, \ldots, n-1$, there are $n - d$ pairs (i, k) satisfying $k - i = d$. Hence,

$$\sum_{i=1}^{n}\sum_{k=1}^{n} |i-k| = 2\sum_{d=1}^{n-1} d(n-d).$$

Therefore,

$$2\sum_{d=1}^{n-1} d(n-d) = 2\left(n\sum_{d=1}^{n-1} d - \sum_{d=1}^{n-1} d^2\right).$$

Using

$$\sum_{d=1}^{n-1} d = \frac{n(n-1)}{2}$$

and

$$\sum_{d=1}^{n-1} d^2 = \frac{n(n-1)(2n-1)}{6},$$

we obtain

$$\sum_{i=1}^{n}\sum_{k=1}^{n} |i-k| = \frac{n(n^2-1)}{3}.$$

□

Remark 1.3. *The function $D_n(v)$ allows the complete work-done to be expressed in the compact form*

$$w^{\varphi}(\alpha) = \sum_{j=1}^{n} D_n(\widehat{\alpha}(j)).$$

This representation will be used repeatedly in deriving the total, average, and extremal values of w^{φ}.

2. Main results

In this section, we establish closed formulae for the total and average complete work-done on the full transformation semigroup T_n, the partial transformation semigroup P_n, and the symmetric inverse semigroup I_n. The results are derived using elementary counting arguments and the distance identity established in Lemma 1.2.

Theorem 2.1. *Let T_n be the full transformation semigroup on $[n]$. For each $\alpha \in T_n$, define*

$$w^{\varphi}(\alpha) = \sum_{i=1}^{n}\sum_{j=1}^{n}\big|i - \alpha(j)\big|.$$

Then

$$W^{\varphi}(T_n) = \sum_{\alpha \in T_n} w^{\varphi}(\alpha) = \frac{n^{\,n+1}(n^2-1)}{3}.$$

Consequently,

$$\overline{W^{\varphi}}(T_n) = \frac{1}{|T_n|}\sum_{\alpha \in T_n} w^{\varphi}(\alpha) = \frac{n(n^2-1)}{3}.$$

Proof. We change the order of summation:

$$\sum_{\alpha \in T_n} w^{\varphi}(\alpha) = \sum_{\alpha \in T_n}\sum_{i=1}^{n}\sum_{j=1}^{n}|i-\alpha(j)| = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\alpha \in T_n}|i-\alpha(j)|.$$

Fix $i, j \in [n]$. For each $k \in [n]$, the number of transformations $\alpha \in T_n$ satisfying $\alpha(j) = k$ is n^{n-1}, since the remaining $n-1$ images may be chosen freely. Hence,

$$\sum_{\alpha \in T_n}|i-\alpha(j)| = n^{n-1}\sum_{k=1}^{n}|i-k|.$$

Therefore,

$$\sum_{\alpha \in T_n} w^{\varphi}(\alpha) = \sum_{i=1}^{n}\sum_{j=1}^{n} n^{n-1}\sum_{k=1}^{n}|i-k|.$$

Since the summation over j contributes a factor of n, we obtain

$$\sum_{\alpha \in T_n} w^{\varphi}(\alpha) = n^n\sum_{i=1}^{n}\sum_{k=1}^{n}|i-k|.$$

By Lemma 1.2,

$$\sum_{i=1}^{n}\sum_{k=1}^{n}|i-k| = \frac{n(n^2-1)}{3}.$$

Thus,

$$W^{\varphi}(T_n) = n^n \cdot \frac{n(n^2-1)}{3} = \frac{n^{\,n+1}(n^2-1)}{3}.$$

Since $|T_n| = n^n$, the average value is

$$\overline{W^{\varphi}}(T_n) = \frac{W^{\varphi}(T_n)}{|T_n|} = \frac{n(n^2-1)}{3}.$$

□

Example 2.2. For $n = 2$,

$$W^{\varphi}(T_2) = \frac{2^3(2^2-1)}{3} = 8, \qquad \overline{W^{\varphi}}(T_2) = 2.$$

For $n = 3$,

$$W^{\varphi}(T_3) = \frac{3^4(3^2-1)}{3} = 216, \qquad \overline{W^{\varphi}}(T_3) = 8.$$

Theorem 2.3. *Let P_n be the partial transformation semigroup on $[n]$. For each $\alpha \in P_n$, let*

$$\widehat{\alpha} : [n] \to \{0, 1, \ldots, n\}$$

be defined by

$$\widehat{\alpha}(j) = \begin{cases} \alpha(j), & j \in \mathrm{dom}(\alpha), \\ 0, & j \notin \mathrm{dom}(\alpha). \end{cases}$$

Define

$$w^{\varphi}(\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} \big| i - \widehat{\alpha}(j) \big|.$$

Then

$$W^{\varphi}(P_n) = \sum_{\alpha \in P_n} w^{\varphi}(\alpha) = \frac{n^2(2n+1)(n+1)^n}{6}.$$

Consequently,

$$\overline{W^{\varphi}}(P_n) = \frac{1}{|P_n|} \sum_{\alpha \in P_n} w^{\varphi}(\alpha) = \frac{n^2(2n+1)}{6}.$$

Proof. Every partial transformation $\alpha \in P_n$ is uniquely represented by the n-tuple

$$\big(\widehat{\alpha}(1), \widehat{\alpha}(2), \ldots, \widehat{\alpha}(n)\big) \in \{0, 1, \ldots, n\}^n.$$

Hence,

$$|P_n| = (n+1)^n.$$

Now,

$$\sum_{\alpha \in P_n} w^{\varphi}(\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\alpha \in P_n} \big| i - \widehat{\alpha}(j) \big|.$$

Fix $i, j \in [n]$. For each $v \in \{0, 1, \ldots, n\}$, the number of elements $\alpha \in P_n$ satisfying $\widehat{\alpha}(j) = v$ is $(n+1)^{n-1}$, since the remaining $n-1$ coordinates are arbitrary. Thus,

$$\sum_{\alpha \in P_n} \big| i - \widehat{\alpha}(j) \big| = (n+1)^{n-1} \sum_{v=0}^{n} |i - v|.$$

Therefore,

$$\sum_{\alpha \in P_n} w^{\varphi}(\alpha) = n(n+1)^{n-1} \sum_{i=1}^{n} \sum_{v=0}^{n} |i - v|.$$

We split the inner summation into the contribution from $v = 0$ and from $v \in [n]$:

$$\sum_{i=1}^{n} \sum_{v=0}^{n} |i - v| = \sum_{i=1}^{n} |i - 0| + \sum_{i=1}^{n} \sum_{k=1}^{n} |i - k|.$$

Now,

$$\sum_{i=1}^{n} |i - 0| = \sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$

and by Lemma 1.2,

$$\sum_{i=1}^{n} \sum_{k=1}^{n} |i - k| = \frac{n(n^2-1)}{3}.$$

Hence,

$$\sum_{\alpha \in P_n} w^{\varphi}(\alpha) = n(n+1)^{n-1} \left(\frac{n(n+1)}{2} + \frac{n(n^2-1)}{3} \right).$$

Simplifying,

$$\sum_{\alpha\in P_n} w^{\varphi}(\alpha) = n^2(n+1)^{n-1}\left(\frac{n+1}{2}+\frac{n^2-1}{3}\right).$$

Since

$$\frac{n+1}{2}+\frac{n^2-1}{3} = \frac{3(n+1)+2(n^2-1)}{6} = \frac{2n^2+3n+1}{6} = \frac{(2n+1)(n+1)}{6},$$

we obtain

$$W^{\varphi}(P_n) = \frac{n^2(2n+1)(n+1)^n}{6}.$$

Dividing by $|P_n| = (n+1)^n$ gives

$$\overline{W^{\varphi}}(P_n) = \frac{n^2(2n+1)}{6}.$$

$\square$

Example 2.4. For $n=2$,

$$W^{\varphi}(P_2) = \frac{2^2(2\cdot 2+1)(2+1)^2}{6} = \frac{4\cdot 5\cdot 9}{6} = 30.$$

Since $|P_2| = 3^2 = 9$, we have

$$\overline{W^{\varphi}}(P_2) = \frac{30}{9} = \frac{10}{3}.$$

Theorem 2.5. *Let I_n be the symmetric inverse semigroup on $[n]$. For each $\alpha\in I_n$, extend α to*

$$\widehat{\alpha} : [n] \to \{0,1,\ldots,n\}$$

by assigning $\widehat{\alpha}(j) = 0$ whenever $j \notin \mathrm{dom}(\alpha)$. Then

$$|I_n| = \sum_{r=0}^{n}\binom{n}{r}^2 r!, \qquad |I_{n-1}| = \sum_{r=0}^{n-1}\binom{n-1}{r}^2 r!.$$

Moreover,

$$W^{\varphi}(I_n) = \sum_{\alpha\in I_n} w^{\varphi}(\alpha) = n^2\left[\frac{n+1}{2}\big(|I_n| - n|I_{n-1}|\big) + \frac{n^2-1}{3}|I_{n-1}|\right].$$

Consequently,

$$\overline{W^{\varphi}}(I_n) = n^2\left[\frac{n+1}{2}\left(1-\frac{n|I_{n-1}|}{|I_n|}\right) + \frac{n^2-1}{3}\frac{|I_{n-1}|}{|I_n|}\right].$$

Proof. Fix $j\in[n]$. We count the elements of I_n according to the value of $\widehat{\alpha}(j)$.

First, suppose that $\widehat{\alpha}(j) = k$, where $k\in[n]$. Then $j\in\mathrm{dom}(\alpha)$ and $\alpha(j) = k$. To form such a partial one-to-one transformation, choose r further domain elements from $[n]\setminus\{j\}$, choose r further image elements from $[n]\setminus\{k\}$, and then choose a bijection between them. Therefore, for each fixed $k\in[n]$,

$$\#\{\alpha\in I_n : \widehat{\alpha}(j) = k\} = \sum_{r=0}^{n-1}\binom{n-1}{r}^2 r! = |I_{n-1}|.$$

Since the possible values of $\widehat{\alpha}(j)$ are $0,1,\ldots,n$, it follows that

$$\#\{\alpha\in I_n : \widehat{\alpha}(j) = 0\} = |I_n| - n|I_{n-1}|.$$

Now,

$$\sum_{\alpha\in I_n} w^{\varphi}(\alpha) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\alpha\in I_n}\big|i-\widehat{\alpha}(j)\big|.$$

Using the preceding counts, we get

$$\sum_{\alpha \in I_n} \big|i - \widehat{\alpha}(j)\big| = \big(|I_n| - n|I_{n-1}|\big)|i - 0| + |I_{n-1}| \sum_{k=1}^{n} |i - k|.$$

Therefore,

$$\sum_{\alpha \in I_n} w^{\varphi}(\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\big(|I_n| - n|I_{n-1}|\big) i + |I_{n-1}| \sum_{k=1}^{n} |i - k| \right].$$

Since the summation over j contributes a factor of n, we obtain

$$\sum_{\alpha \in I_n} w^{\varphi}(\alpha) = n \left[\big(|I_n| - n|I_{n-1}|\big) \sum_{i=1}^{n} i + |I_{n-1}| \sum_{i=1}^{n} \sum_{k=1}^{n} |i - k| \right].$$

Using

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

and Lemma 1.2,

$$\sum_{i=1}^{n} \sum_{k=1}^{n} |i - k| = \frac{n(n^2 - 1)}{3},$$

we have

$$\sum_{\alpha \in I_n} w^{\varphi}(\alpha) = n \left[\big(|I_n| - n|I_{n-1}|\big) \frac{n(n+1)}{2} + |I_{n-1}| \frac{n(n^2 - 1)}{3} \right].$$

Thus,

$$W^{\varphi}(I_n) = n^2 \left[\frac{n+1}{2} \big(|I_n| - n|I_{n-1}|\big) + \frac{n^2 - 1}{3} |I_{n-1}| \right].$$

Dividing by $|I_n|$ gives the required average. $\square$

Example 2.6. For $n = 2$, we have

$$|I_2| = \sum_{r=0}^{2} \binom{2}{r}^2 r! = 1 + 4 + 2 = 7$$

and

$$|I_1| = \sum_{r=0}^{1} \binom{1}{r}^2 r! = 1 + 1 = 2.$$

Hence,

$$W^{\varphi}(I_2) = 2^2 \left[\frac{3}{2}(7 - 2 \cdot 2) + \frac{3}{3} \cdot 2 \right].$$

Therefore,

$$W^{\varphi}(I_2) = 4 \left[\frac{3}{2}(3) + 2 \right] = 4 \left(\frac{9}{2} + 2 \right) = 26.$$

Thus,

$$\overline{W^{\varphi}}(I_2) = \frac{26}{7}.$$

3. Extremal values

In this section, we determine the minimum and maximum values of w^φ on T_n and P_n. The proofs rely on the distance function

$$D_n(v) = \sum_{i=1}^{n} |i - v|.$$

Theorem 3.1. *Let T_n be the full transformation semigroup on $[n]$. For $\alpha \in T_n$, define*

$$w^\varphi(\alpha) = \sum_{i=1}^{n}\sum_{j=1}^{n} |i - \alpha(j)|.$$

Then

$$w^\varphi(\alpha) = \sum_{j=1}^{n} D_n(\alpha(j)).$$

Moreover,

$$\min_{\alpha \in T_n} w^\varphi(\alpha) = \begin{cases} nm^2, & n = 2m, \\ nm(m+1), & n = 2m+1, \end{cases}$$

and

$$\max_{\alpha \in T_n} w^\varphi(\alpha) = \frac{n^2(n-1)}{2}.$$

The minimum is attained by constant transformations whose values are medians of $[n]$. The maximum is attained by transformations satisfying $\alpha(j) \in \{1, n\}$ for all $j \in [n]$.

Proof. For each fixed $j \in [n]$,

$$\sum_{i=1}^{n} |i - \alpha(j)| = D_n(\alpha(j)).$$

Therefore,

$$w^\varphi(\alpha) = \sum_{j=1}^{n} D_n(\alpha(j)).$$

Since the terms $D_n(\alpha(j))$ are independent, the minimum and maximum of w^φ are obtained by choosing each value $\alpha(j)$ to minimize or maximize $D_n(k)$.

For $k \in [n]$,

$$D_n(k) = \sum_{i=1}^{k-1}(k-i) + \sum_{i=k+1}^{n}(i-k).$$

Hence,

$$D_n(k) = \frac{(k-1)k}{2} + \frac{(n-k)(n-k+1)}{2}.$$

This quadratic expression is minimized at the median values of $[n]$ and maximized at the endpoints 1 and n.

If $n = 2m$, then

$$D_n(m) = D_n(m+1) = m^2.$$

If $n = 2m+1$, then

$$D_n(m+1) = m(m+1).$$

Furthermore,

$$D_n(1) = D_n(n) = \frac{n(n-1)}{2}.$$

Multiplying these values by n, since there are n choices of j, gives the stated extremal values. □

Theorem 3.2. *Let P_n be the partial transformation semigroup on $[n]$. For $\alpha \in P_n$, define*

$$w^{\varphi}(\alpha) = \sum_{i=1}^{n}\sum_{j=1}^{n}\big|i - \widehat{\alpha}(j)\big|.$$

Then

$$w^{\varphi}(\alpha) = \sum_{j=1}^{n} D_n(\widehat{\alpha}(j)).$$

Moreover,

$$\min_{\alpha\in P_n} w^{\varphi}(\alpha) = \begin{cases} nm^2, & n = 2m, \\ nm(m+1), & n = 2m+1, \end{cases}$$

and

$$\max_{\alpha\in P_n} w^{\varphi}(\alpha) = \frac{n^2(n+1)}{2}.$$

The minimum is attained by median-valued constant total transformations, while the maximum is attained by the nowhere-defined transformation.

Proof. By definition,

$$w^{\varphi}(\alpha) = \sum_{i=1}^{n}\sum_{j=1}^{n}|i - \widehat{\alpha}(j)|.$$

Fixing j, we obtain

$$\sum_{i=1}^{n}|i - \widehat{\alpha}(j)| = D_n(\widehat{\alpha}(j)).$$

Thus,

$$w^{\varphi}(\alpha) = \sum_{j=1}^{n} D_n(\widehat{\alpha}(j)).$$

For $v = 0$,

$$D_n(0) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

For $v \in [n]$, the function $D_n(v)$ has the same form as in Theorem 3.1. Therefore, its minimum occurs at the median values of $[n]$. Since

$$D_n(0) = \frac{n(n+1)}{2} > \frac{n(n-1)}{2} = D_n(1) = D_n(n),$$

the maximum over $v \in \{0, 1, \dots, n\}$ occurs at $v = 0$. Hence, the maximum of w^{φ} on P_n is obtained when

$$\widehat{\alpha}(j) = 0 \qquad \text{for all } j \in [n],$$

that is, when α is the nowhere-defined transformation. Multiplying the corresponding extremal values of $D_n(v)$ by n gives the stated result. □

4. Comparison of average complete work-done

The average complete work-done on P_n is always greater than the corresponding value on T_n. This follows from the closed formulae obtained above.

Theorem 4.1. *For every $n \geq 1$,*

$$\overline{W^{\varphi}}(P_n) - \overline{W^{\varphi}}(T_n) = \frac{n(n+2)}{6}.$$

In particular,

$$\overline{W^{\varphi}}(P_n) > \overline{W^{\varphi}}(T_n)$$

for all $n \geq 1$.

Proof. From Theorems 2.1 and 2.3,

$$\overline{W^{\varphi}}(T_n) = \frac{n(n^2-1)}{3}$$

and

$$\overline{W^{\varphi}}(P_n) = \frac{n^2(2n+1)}{6}.$$

Therefore,

$$\overline{W^{\varphi}}(P_n) - \overline{W^{\varphi}}(T_n) = \frac{n^2(2n+1)}{6} - \frac{n(n^2-1)}{3}.$$

Thus,

$$\overline{W^{\varphi}}(P_n) - \overline{W^{\varphi}}(T_n) = \frac{2n^3+n^2-2n^3+2n}{6} = \frac{n^2+2n}{6}.$$

Hence,

$$\overline{W^{\varphi}}(P_n) - \overline{W^{\varphi}}(T_n) = \frac{n(n+2)}{6} > 0$$

for every $n \geq 1$. □

5. Numerical results and discussion

n	$\lvert T_n\rvert = n^n$	$w^{\varphi}(T_n)$	$\overline{W^{\varphi}}(T_n)$
1	1	0	0
2	4	8	2
3	27	216	8
4	256	5 120	20
5	3 125	125 000	40
6	46 656	3 265 920	70
7	823 543	92 236 816	112
8	16 777 216	2 818 572 288	168
9	387 420 489	92 980 917 360	240
10	10 000 000 000	3 300 000 000 000	330

Table 1. Numerical values for the full transformation semigroup T_n.

n	$\lvert P_n\rvert = (n+1)^n$	$w^{\varphi}(P_n)$	$\overline{W^{\varphi}}(P_n)$
1	2	1	$\frac{1}{2}$
2	9	30	$\frac{10}{3}$
3	64	672	$\frac{21}{2}$
4	625	15 000	24
5	7 776	356 400	$\frac{275}{6}$
6	117 649	9 176 622	78
7	2 097 152	256 901 120	$\frac{245}{2}$
8	43 046 721	7 805 805 408	$\frac{544}{3}$
9	1 000 000 000	256 500 000 000	$\frac{513}{2}$
10	25 937 424 601	9 078 098 610 350	350

Table 2. Numerical values for the partial transformation semigroup P_n, where undefined values are represented by 0.

n	$\lvert I_n\rvert = \sum_{r=0}^{n} \binom{n}{r}^2 r!$	$w^{\varphi}(I_n)$	$\overline{W^{\varphi}}(I_n)$
1	2	1	0.500000
2	7	26	3.714286
3	34	402	11.823529
4	209	5 640	26.985646
5	1 546	79 375	51.342173
6	13 327	1 159 746	87.022286
7	130 922	17 824 436	136.145461
8	1 441 729	289 532 832	200.823339
9	17 572 114	4 975 738 605	283.161070
10	234 662 231	90 405 576 250	385.258317

TABLE 3. Numerical values for the symmetric inverse semigroup I_n, where undefined values are represented by 0.

Tables 1–3 show that the total value of w^{φ} increases rapidly as n increases. This is expected because the number of transformations grows significantly with n. The full transformation semigroup T_n has average value

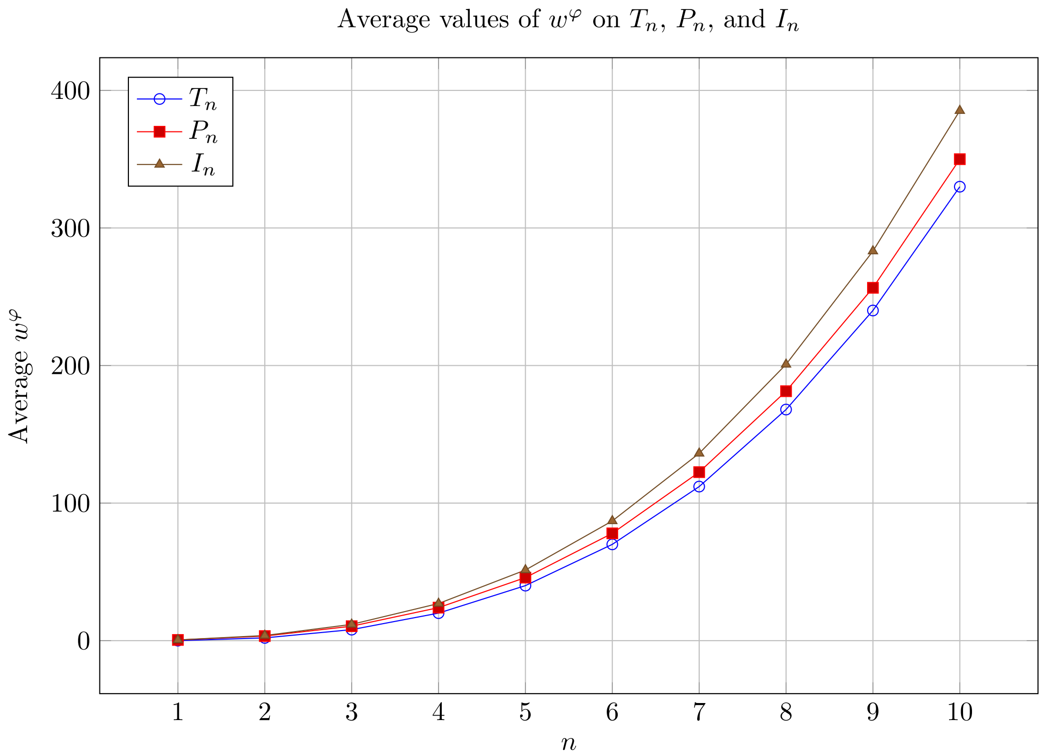
$$\overline{W^{\varphi}}(T_n) = \frac{n(n^2-1)}{3},$$

while the partial transformation semigroup P_n has average value

$$\overline{W^{\varphi}}(P_n) = \frac{n^2(2n+1)}{6}.$$

The latter is larger because undefined values are represented by 0, thereby increasing the displacement through terms of the form $|i-0|$.

For the symmetric inverse semigroup I_n, the injectivity condition prevents repetition of image values. Although this restriction reduces the number of admissible transformations, it tends to distribute image values more evenly. This explains why the average values for I_n are larger than those for T_n and P_n in the displayed numerical range.



The graph illustrates the increasing trend of the average value of w^φ for all three semigroups. For the computed values,

$$\mathrm{Avg}(T_n) < \mathrm{Avg}(P_n) < \mathrm{Avg}(I_n), \qquad 1 \le n \le 10.$$

The curve for T_n is the lowest, reflecting the unrestricted possibility of repeated images. The curve for P_n lies above that of T_n because undefined values contribute additional distance from 0. The curve for I_n is the highest in the displayed range, reflecting the effect of injectivity on the distribution of image values.

6. Conclusion

This paper has obtained closed formulae for the total and average values of the complete work-done function w^φ on T_n, P_n, and I_n. The results show that the algebraic structure of a transformation semigroup has a direct influence on the magnitude of the complete work-done values. In particular, partiality increases the average value by introducing the value 0, while injectivity changes the distribution of images and leads to larger average values in the numerical range considered.

The extremal results for T_n and P_n further demonstrate that the minimum values are attained at median-valued constant transformations, while the maximum values occur at endpoint values for T_n and at the nowhere-defined transformation for P_n. These results provide a basis for further studies of distance-based and work-done functions on other finite semigroups and related algebraic structures.

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