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# Development and Implementation of a Numerical method to Solve Third-Order Initial Value Problems Using an Optimized Hybrid Volterra Integral Equation of the Second Kind

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## Abstract:

This paper introduced the formulation and execution of a numerical technique for addressing third-order initial value issues through an optimized hybrid Volterra integral equation of the second sort. Power series and exponential fitting serve as basis functions for the development of a novel two-step optimized hybrid numerical approach, suitable for addressing stiff initial value problems in third-order ordinary differential equations. The novel method exhibits enhanced convergence characteristics and has demonstrated its effectiveness on benchmark issues. The numerical implementation exhibits diminished computing expense, increased precision and superior stability characteristics relative to conventional approaches. The optimized hybrid block designs have superior stability qualities. Numerical examples are provided to demonstrate the dependability and precision of the approximations.

## Keywords:

Two-step; exponentially block approach; Volterra integral equation, Optimized Hybrid Method, Stiff ODEs.

## I. Introduction

This study detailed the creation and implementation of a numerical technique for addressing third-order initial value issues through an optimized hybrid Volterra integral equation of the second sort. Power series and exponential fitting serve as the foundational basis functions for deriving a novel two-step optimized hybrid numerical approach, suitable for addressing stiff initial value problems in third-order ordinary differential equations. The second type of Volterra Integral Equation (VIE) is expressed in the following form:

$$y(x) = f(x) + \int_{\alpha(x)}^{\beta(x)} k(x,s)y(s)\,ds, \qquad (1.1)$$

This work reformulates (1.1) into a third-order Initial Value Problem (IVP) for a hybrid Volterra Integral Equation (VIE) of the following form

$$y'''(x) = f'''(x) + \int_{\alpha(x)}^{\beta(x)} K\left(x, s, y(s) y'(s), y''(s)\right) ds, \quad x \in [x_0, X] \qquad (1.2)$$

It is evident that solving (1.2) is synonymous with addressing the initial value problems of third-order Ordinary Differential Equations (ODEs).

$$y'''(x) = f'''(x) + \varphi\left(x, y(x), y'(x), y''(x)\right), \quad y(x_0) = f(x_0), \; y'(x_0) = f'(x_0), \; y''(x_0) = f''(x_0), \qquad (1.3)$$

Consequently, the resolution of the integral equation (1.2) and the initial-value problem (1.3) can be achieved using a singular technique. Transforming (1.2) into a system of first-order equations prior to employing an approximate method for problem-solving frequently results in increased processing expense. It was believed before now that higher order differential equations can be solved only when they are reduced to first order which were discussed in Anake (2011) Chan et al. (2004), Gholamtabar and Parandin (2014), Akinfenwa et al. (2018). One method for solving higher-order ordinary differential equations is the direct application of the Predictor-Corrector method, extensively discussed in Kayode and Adeyeye (2013), Adesanya et al. (2008), Awoyemi and Idowu (2005), Jator (2007), and Jator and Li (2009). These authors developed linear multistep methods with continuous coefficients, which offer the advantage of evaluation at all points within the grid, in contrast to the method proposed by Awoyemi et al. (2011) and Adeniran and Ogundare (2015), who suggested using Chebyshev series as the fundamental function for generating interpolation and collocation equations in the development of continuous hybrid linear multistep methods. Anake et al. (2012) proposed a continuous implicit one-step method with zero stability for solving initial value problems of second-order differential equations, as noted by Adesanya and Anake (2008) and Adesanya et al. (2014). James *et al.* (2013), Awoyemi (2001),Fatunla (1991), Awoyemi *et al.* (2009), Abdelrahim and Omar (2016) to mentioned but a few. Hybrid methods combine two or more different numerical techniques, for example an explicit method (such as Runge-Kutta) might be used for predicting the solution at the next time step, while an implicit method (such as backward differentiation) could be used for refining or correcting that prediction. According to Raymond et al. (2023), the approach of developing the new optimized numerical scheme is by adding the off-grid points to the usual k-step scheme.

## II Methodology

### A. Derivation of the Novel Two-Step Hybrid Optimized Volterra Integral Equation of the Second Kind with Dual Optimization Points.

For the resolution of (1.2), we propose utilizing a combination of power series and an exponential fitted function as the approximate solution, with the approach structured as follows:

$$\varphi'''(x) = \phi'''(x) + \sum_{j=0}^{s} \mu_i x^j + \sum_{j=t}^{q} \psi_j e^{x^j} \qquad (1.4)$$

Be the approximate solution of (1.2) where $\mu_i$ and $\psi_i$ are the coefficients to be determined.

Let the approximate power series and exponentially fitted function be of the form

$$\sum_{i=0}^{k} \tau_i \varphi_{n+i} = \sum_{i=0}^{k} \tau_i \phi_{n+i} + h^3 \sum_{i=0}^{s} \varsigma_i (t) \mu_{n+i} + h^3 \sum_{j=i}^{r} \psi_j (t) e^{x^j} \qquad (1.5)$$

$$\sum_{i=0}^{k} \tau_i \left( \varphi'''_{n+i} - \phi'''_{n+i} \right) = h^3 \left( \sum_{j=0}^{s} \varsigma_j (t) \mu_{n+j} + \sum_{j=i}^{r} \psi_j (t) e^{x^j} \right) \qquad (1.6)$$

where $\tau_i, \varsigma_j . \psi_j, (i=0,1,\cdots,m;\ j=i,\cdots,k)$ are constant coefficient of (1.6) are to be calculated, as they are regarded as the solution to the second Volterra integral equation. To formulate this strategy, two off-grid points will be provided, positioned between $x_n$ and $x_{n+1}$ ,the corresponding coordinates. In the two-step procedure, the value of k will be set to two; hence, we will optimize $u$ and $v$ .and.
From (1.6), we derive

$$\varphi'''(x) = \phi'''(x) + \sum_{j=0}^{3} \mu_i x^j + \sum_{j=1}^{4} \psi_j e^{x^j} \quad (1.7)$$

By utilizing the first, second, and third derivatives of equation (1.7), we derive

$$\varphi'(x) - \phi'(x) = \sum_{j=0}^{3} jx^{j-1}\mu_j(x) + \sum_{j=1}^{4} jx^{j-1}\psi_j(x) e^{x^j} \quad (1.8)$$

$$\varphi''(x) - \phi''(x) = \sum_{j=0}^{3} j(j-1)x^{j-2}\mu_j(x) + \sum_{j=1}^{4} \psi_j \, j\left[(j-1)x^{j-2} + jx^{2(j-2)}\right] e^{x^j} \quad (1.9)$$

$$\varphi'''(x) - \phi'''(x) = \sum_{j=0}^{3} j(j-1)(j-2)x^{j-3}\mu_j + \sum_{j=1}^{4} jx^{j-3}\psi_j(x) e^{x^j}\left(j^2x^{2j} - 3j - 3jx^j + 3j^2x^j + j^2 + 2\right) \quad (1.10)$$

Interpolating (1.7) - (1.9) at the points $x_n = 0$ and collocating (1.10) at all points $t_{n+m} = t_n + qh, m = \{0, u, v, 1, 2\}$. Equation (1.4) result to system of nonlinear equation of the form

$$MD = V \quad (1.11)$$

$$\begin{bmatrix} 12 & 12x_n & 9x_n^2 & 7x_n^3 & \frac{1}{4}x_n^4 & \frac{1}{20}x_n^5 & \frac{1}{120}x_n^6 & \frac{1}{840}x_n^7 \\ 0 & 12 & 18x_n & 21x_n^2 & x_n^3 & \frac{1}{4}x_n^4 & \frac{1}{20}x_n^5 & \frac{1}{120}x_n^6 \\ 0 & 0 & 18 & 42x_n & 3x_n^2 & 3x_n^3 & \frac{1}{4}x_n^4 & \frac{1}{20}x_n^5 \\ 0 & 0 & 0 & 42 & 6x_n & 3x_n^2 & x_n^3 & \frac{1}{4}x_n^4 \\ 0 & 0 & 0 & 42 & 6(x_n+uh) & 3(x_n+uh)^2 & (x_n+uh)^3 & \frac{1}{4}(x_n+uh)^4 \\ 0 & 0 & 0 & 42 & 6(x_n+vh) & 3(x_n+vh)^2 & (x_n+vh)^3 & \frac{1}{4}(x_n+vh)^4 \\ 0 & 0 & 0 & 42 & 6(x_n+h) & 3(x_n+h)^2 & (x_n+h)^3 & \frac{1}{4}(x_n+h)^4 \\ 0 & 0 & 0 & 42 & 6(x_n+2h) & 3(x_n+2h)^2 & (x_n+2h)^3 & \frac{1}{4}(x_n+2h)^4 \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_0' \\ \mu_0'' \\ \psi_0 \\ \psi_u \\ \psi_v \\ \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} (\varphi_0 - \phi_0) \\ (\varphi_0' - \phi_0') \\ (\varphi_0'' - \phi_0'') \\ g_n \\ g_{n+u} \\ g_{n+v} \\ g_{n+1} \\ g_{n+2} \end{bmatrix}$$

Using Gaussian elimination method on (1.11) gives the coefficients of $\mu_j, \psi_j, j = 0(1)7.$ The values are subsequently swapped into (1.4) to get the implicit continuous optimized hybrid Volterra integral equation of the second sort in the following form;

$$p\left((\varphi_n + \xi h) - (\phi_n + \xi h)\right) = \mu_0(\varphi_n - \phi_n) + h\mu_1(\varphi'_n - \phi'_n) + h^2\mu_2(\varphi''_n - \phi''_n) + h^3\left[\psi_0 g_n + \psi_u g_{n+u} + \psi_v g_{n+v} + \psi_1 g_{n+1} + \psi_2 g_{n+2}\right]$$

(1.12)

Differentiating (1.12) once and twice we obtain

$$hp'\left(\left(\varphi_n+qh\right)-\left(\phi_n+qh\right)\right)=h\mu_0\left(\varphi'_n-\phi'_n\right)+h^2\mu_1\left(\varphi''_n-\phi''_n\right) + h^3\left[\psi_0 g_n+\psi_u g_{n+u}+\psi_v g_{n+v}+\psi_1 g_{n+1}+\psi_2 g_{n+2}\right] \tag{1.13}$$

$$h^2p''\left(\left(\varphi_n+qh\right)-\left(\phi_n+qh\right)\right)=h^2\mu_0\left(\varphi''_n-\phi''_n\right)+h^3\left[\psi_0 g_n+\psi_u g_{n+u}+\psi_v g_{n+v}+\psi_1 g_{n+1}+\psi_2 g_{n+2}\right]$$

(1.14)

Substituting $\xi=1$ in (1.12), we obtain an approximate volterra integral equation to the solution of (1.3) at the points $t_{n+1}$ which yield

(1.3) at the points $t_{n+1}$ which yield

$$p\left(\left(\varphi_n+h\right)-\left(\phi_n+h\right)\right)=\left(\varphi_n-\phi_n\right)+h\mu_1\left(\varphi'_n-\phi'_n\right)+\tfrac{1}{2}h^2\mu_2\left(\varphi''_n-\phi''_n\right)$$
$$+h^3\begin{bmatrix}\frac{1}{1680}\left(\frac{-35u-35v+189uv+11}{uv}\right)g_n-\frac{1}{840}\left(\frac{35v-11}{u(u-1)(u-2)(u-v)}\right)g_{n+u}\\ +\frac{1}{840}\left(\frac{35u+11}{v(v-1)(v-2)(u-v)}\right)g_{n+v}+\frac{1}{840}\left(\frac{-21u-21v+56uv+10}{(v-1)(u-1)}\right)g_{n+1}\\ -\frac{1}{1680}\left(\frac{-7u-7v+21uv+3}{(v-2)(u-2)}\right)g_{n+2}\end{bmatrix}$$

(1.15)

Also Substituting $\xi=1$ in (1.13) we obtain an approximate volterra integral equation to the solution of (1.3) at the points $t'_{n+1}$ which yield

$$hp'\left(\left(\varphi_n+h\right)-\left(\phi_n+h\right)\right)=h\left(\varphi'_n-\phi'_n\right)+h^2\mu_2\left(\varphi''_n-\phi''_n\right)$$
$$+h^2\begin{bmatrix}\frac{1}{120}\left(\frac{-8u-8v+35uv+3}{uv}\right)g_n-\frac{1}{60}\left(\frac{8v-3}{u(u-1)(u-2)(u-v)}\right)g_{n+u}\\ +\frac{1}{60}\left(\frac{8u-3}{v(v-1)(v-2)(u-v)}\right)g_{n+v}+\frac{1}{60}\left(\frac{-7u-7v+15uv+4}{(v-1)(u-1)}\right)g_{n+1}\\ -\frac{1}{120}\left(\frac{-2u-2v+5uv+1}{(v-2)(u-2)}\right)g_{n+2}\end{bmatrix}$$

(1.16)

Finally, substituting $\xi = 1$ in (1.14) we obtain an approximate volterra integral equation to the solution of (1.3) at the points $t'_{n+1}$ which yield

$$h^2 p''\left((\varphi_n + h) - (\phi_n + h)\right) = h^2\left(\varphi''_n - \phi''_n\right) + h^3\left[\begin{array}{l} \frac{1}{120}\left(\frac{-15u-15v+50uv+7}{uv}\right)g_n - \frac{1}{60}\left(\frac{15v-7}{u(u-1)(u-2)(u-v)}\right)g_{n+u} \\ +\frac{1}{60}\left(\frac{15u-3}{v(v-1)(v-2)(u-v)}\right)g_{n+v} + \frac{1}{60}\left(\frac{-25u-25v+40uv+18}{(v-1)(u-1)}\right)g_{n+1} \\ -\frac{1}{120}\left(\frac{-5u-5v+10uv+3}{(v-2)(u-2)}\right)g_{n+2} \end{array}\right]$$

(1.17)

Expanding (1.15) - (1.17) using Taylor series around the points $t_n$ we obtain

$$\left[\begin{array}{l} \sum_{j=0}^{2}\frac{(h^3)}{j!}(\varphi_n-\phi_n)^j - (\varphi_n-\phi_n) - h(\varphi'_n-\phi'_n) - \frac{1}{2}h^2(\varphi''_n-\phi''_n) - \sum_{j=0}^{2}\frac{h^{j+3}}{j!}\left[\begin{array}{l} \frac{1}{1680}\left(\frac{-35u-35v+189uv+11}{uv}\right)g_n - \frac{1}{840}\left(\frac{35v-11}{u(u-1)(u-2)(u-v)}\right)g_{n+u} \\ +\frac{1}{840}\left(\frac{35u+11}{v(v-1)(v-2)(u-v)}\right)g_{n+v} + \frac{1}{840}\left(\frac{-21u-21v+56uv+10}{(v-1)(u-1)}\right)g_{n+1} \\ -\frac{1}{1680}\left(\frac{-7u-7v+21uv+3}{(v-2)(u-2)}\right)g_{n+2} \end{array}\right] \\ \sum_{j=0}^{2}\frac{(h^2)}{j!}(\varphi_n-\phi_n)^j - h(\varphi'_n-\phi'_n) - \frac{1}{2}h^2(\varphi''_n-\phi''_n) - \sum_{j=0}^{2}\frac{h^{j+2}}{j!}\left[\begin{array}{l} \frac{1}{120}\left(\frac{-8u-8v+35uv+3}{uv}\right)g_n - \frac{1}{60}\left(\frac{8v-3}{u(u-1)(u-2)(u-v)}\right)g_{n+u} \\ +\frac{1}{60}\left(\frac{8u-3}{v(v-1)(v-2)(u-v)}\right)g_{n+v} + \frac{1}{60}\left(\frac{-7u-7v+15uv+4}{(v-1)(u-1)}\right)g_{n+1} \\ -\frac{1}{120}\left(\frac{-2u-2v+5uv+1}{(v-2)(u-2)}\right)g_{n+2} \end{array}\right] \\ \sum_{j=0}^{2}\frac{(h)}{j!}(\varphi_n-\phi_n)^j - h^2(\varphi''_n-\phi''_n) - \sum_{j=0}^{2}\frac{h^{j+1}}{j!}\left[\begin{array}{l} \frac{1}{120}\left(\frac{-15u-15v+50uv+7}{uv}\right)g_n - \frac{1}{60}\left(\frac{15v-7}{u(u-1)(u-2)(u-v)}\right)g_{n+u} \\ +\frac{1}{60}\left(\frac{15u-3}{v(v-1)(v-2)(u-v)}\right)g_{n+v} + \frac{1}{60}\left(\frac{-25u-25v+40uv+18}{(v-1)(u-1)}\right)g_{n+1} \\ -\frac{1}{120}\left(\frac{-5u-5v+10uv+3}{(v-2)(u-2)}\right)g_{n+2} \end{array}\right] \end{array}\right] = 0$$

(1.18)

We obtain the corresponding local truncation error from (1.18) as

$$L\left[\varphi\left(t_{n+1}\right); h\right] = \frac{1}{16800}(-22u-22v+70uv+9)$$

(1.19)

$$L\left[\varphi'\left(t_{n+1}\right);h\right]=\frac{1}{4200}(-21u-21v+56uv+10) \quad (1.20)$$

$$L\left[\varphi''\left(t_{n+1}\right);h\right]=\frac{1}{600}(-7u-7v+15uv+4) \quad (1.21)$$

Setting the primary terms of the local truncation errors in (1.19) – (1.21) to zero .Solving (1.20) and (1.21) simultaneously, we obtain an optimized value of $u$ and $v$ as

$$v=\frac{29}{64}+\frac{1}{64}\sqrt{201}=\frac{13493}{20000},u=\frac{29}{64}-\frac{1}{64}\sqrt{201}=\frac{579}{2500} \quad (1.22)$$

We then Substitute these optimized values of $u$ and $v$ in (1.11) to obtain.

$$\begin{bmatrix} 12 & 12x_n & 9x_n^2 & 7x_n^3 & \frac{1}{4}x_n^4 & \frac{1}{20}x_n^5 & \frac{1}{120}x_n^6 & \frac{1}{840}x_n^7 \\ 0 & 12 & 18x_n & 21x_n^2 & x_n^3 & \frac{1}{4}x_n^4 & \frac{1}{20}x_n^5 & \frac{1}{120}x_n^6 \\ 0 & 0 & 18 & 42x_n & 3x_n^2 & 3x_n^3 & \frac{1}{4}x_n^4 & \frac{1}{20}x_n^5 \\ 0 & 0 & 0 & 42 & 6x_n & 3x_n^2 & x^3{}_n & \frac{1}{4}x_n^4 \\ 0 & 0 & 0 & 42 & 6(x_n+\frac{579}{2500}h) & 3(x_n+\frac{579}{2500}h)^2 & (x_n+\frac{579}{2500}h)^3 & \frac{1}{4}(x_n+\frac{579}{2500}h)^4 \\ 0 & 0 & 0 & 42 & 6(x_n+\frac{13493}{20000}h) & 3(x_n+\frac{13493}{20000}h)^2 & (x_n+\frac{13493}{20000}h)^3 & \frac{1}{4}(x_n+\frac{13493}{20000}h)^4 \\ 0 & 0 & 0 & 42 & 6(x_n+h) & 3(x_n+h)^2 & (x_n+h)^3 & \frac{1}{4}(x_n+h)^4 \\ 0 & 0 & 0 & 42 & 6(x_n+2h) & 3(x_n+2h)^2 & (x_n+2h)^3 & \frac{1}{4}(x_n+2h)^4 \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_0' \\ \mu_0'' \\ \psi_0 \\ \psi_{\frac{579}{2500}} \\ \psi_{\frac{13493}{20000}} \\ \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} (\varphi_0-\phi_0) \\ (\varphi_0'-\phi_0') \\ (\varphi_0''-\phi_0'') \\ g_n \\ g_{n+\frac{579}{2500}} \\ g_{n+\frac{13493}{20000}} \\ g_{n+1} \\ g_{n+2} \end{bmatrix}$$

(1.23)

Using Gaussian elimination method on (1.23) gives the coefficients of $\mu_0, \mu'_0, \mu''_0, \psi_0, \psi_{\frac{579}{2500}}, \psi_{\frac{13493}{20000}}, \psi_1, \psi_2$ .These values are then substituted into (1.4) to give the implicit continuous optimized hybrid Volterra integral equation of the second kind scheme of the form;

$$p\left(\left(\varphi_n+\xi h\right)-\left(\phi_n+\xi h\right)\right)=\mu_0\left(\varphi_n-\phi_n\right)+h\mu'_0\left(\varphi'_n-\phi'_n\right)+h^2\mu''_0\left(\varphi''_n-\phi''_n\right)$$
$$+h^3\left[\psi_0 g_n+\psi_{\frac{579}{2500}}g_{n+\frac{579}{2500}}+\psi_{\frac{13493}{20000}}g_{n+\frac{13493}{20000}}+\psi_1 g_{n+1}+\psi_2 g_{n+2}\right],\ \xi=\left\{\frac{579}{2500},\frac{13493}{20000},1,2\right\}$$

(1.24)

Differentiating (1.24) first and second time and evaluating at the points $t_{n+\psi} = t_n + \psi h,\ \psi = \left\{0, \frac{576}{2500}, \frac{13493}{20000}, 1, 2\right\}$ then substituting into (1.25), We derive the discrete optimized hybrid Volterra integral equation in the following form.

$$A^{(0)}B_m^{[1]} = A^{(1)}C_m^{[0]} + \sum_{i=0} D^{[i]}G_m^{[i]} + \sum_{j=u,v,1,2} D^{[j]}G_m^{[j]} \qquad (1.25)$$

Where

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B_m^{[1]} = \begin{bmatrix} \left(\varphi_{n+\frac{579}{2500}} - \phi_{n+\frac{579}{2500}}\right) \\ \left(\varphi_{n+\frac{13493}{20000}} - \phi_{n+\frac{13493}{20000}}\right) \\ \left(\varphi_{n+1} - \phi_{n+1}\right) \\ \left(\varphi_{n+2} - \phi_{n+2}\right) \\ \left(\varphi'_{n+\frac{579}{2500}} - \phi'_{n+\frac{579}{2500}}\right) \\ \left(\varphi'_{n+\frac{13493}{20000}} - \phi'_{n+\frac{13493}{20000}}\right) \\ \left(\varphi'_{n+1} - \phi'_{n+1}\right) \\ \left(\varphi'_{n+2} - \phi'_{n+2}\right) \\ \left(\varphi''_{n+\frac{579}{2500}} - \phi''_{n+\frac{579}{2500}}\right) \\ \left(\varphi''_{n+\frac{13493}{20000}} - \phi''_{n+\frac{13493}{20000}}\right) \\ \left(\varphi''_{n+2} - \phi''_{n+2}\right) \\ \left(\varphi''_{n+2} - \phi''_{n+2}\right) \end{bmatrix}$$

$$A^{(1)}=\begin{bmatrix} 1 & \frac{579}{2500}h & \frac{335241}{12500000}h^2 \\ 1 & \frac{13493}{20000}h & \frac{182061049}{800000000}h^2 \\ 1 & h & \frac{1}{2}h^2 \\ 1 & 2h & 2h^2 \\ 0 & 1 & \frac{579}{2500}h \\ 0 & 1 & \frac{13493}{20000}h \\ 0 & 1 & h \\ 0 & 1 & 2h \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} C_m^{[0]}=\begin{bmatrix} (\varphi_n-\phi_n) \\ (\varphi'_n-\phi'_n) \\ (\varphi''_n-\phi''_n) \end{bmatrix}, D^{[i]}=\begin{bmatrix} \frac{19968192143515908 2763}{14757968750000000000000}h^3 \\ \frac{25982971258897599261393859}{166010880000000000000000000}h^3 \\ \frac{146871661}{4374970320}h^3 \\ \frac{57809161}{820306935}h^3 \\ \frac{158597272523530029}{1054140625000000000}h^2 \\ \frac{4090459695927236804171}{88934400000000000000000}h^2 \\ \frac{12187129}{187498728}h^2 \\ -\frac{2500106}{23437341}h^2 \\ \frac{774600153924897}{8433125000000000}h \\ \frac{231937300020434917}{444672000000000000}h \\ \frac{6093485}{93749364}h \\ -\frac{12187553}{23437341}h \end{bmatrix}, G_m^{[i]}=[g_n]$$

$$D^{[j]}=\begin{bmatrix} \frac{2871798838999390 01259h^3}{3292370287543750000000000} & -\frac{1498150282390990056936h^3}{6526449380659371337890625} & \frac{1940912887559953 35249h^3}{2531817968750000000000000} & -\frac{2519064522795079523223h^3}{12817377015625000000000000} \\ \frac{174113154862014111346786039 6957h^3}{499720492544906035200000000000000} & \frac{849140035860136382538191h^3}{256763568631752000000000000000} & \frac{198151574984717543171381 69539h^3}{537597720576000000000000000000000} & -\frac{147684785276254666071183575 39h^3}{1007999544115200000000000000000000} \\ \frac{32845703125000h^3}{305005183438053} & \frac{11576000000000000h^3}{433063853943528717} & -\frac{3516367h^3}{2624988870} & \frac{3126113h^3}{196874910960} \\ \frac{40037500000000h^3}{53824444136127} & \frac{48486400000000000h^3}{433063853943528717} & \frac{30514444h^3}{7705555} & \frac{145312871h^3}{12304681935} \\ \frac{26259788815091061h^2}{188135445002500000 0} & -\frac{4808499816880527424h^2}{1491759858436427734375} & \frac{19290386785432701 7h^2}{1808441406250000000000} & -\frac{248298429838254570 9h^2}{915526929687500000000000} \\ \frac{849504962657530852893406 49h^2}{535414813440970752000000000} & \frac{3203099022505028 2303h^2}{122268366015120000000 0} & -\frac{10604372837149886380531 01h^2}{31999864320000000000000000} & \frac{6449626710657056296393 03h^2}{179999918592000000000000000} \\ \frac{37456250000000h^2}{130716507187737} & \frac{3059200000000000h^2}{20622088283025177} & -\frac{53h^2}{49999788} & \frac{312553h^2}{2812498728} \\ \frac{168662500000000h^2}{130716507187737} & -\frac{9881600000000000h^2}{20622088283025177} & \frac{15416596h^2}{12499947} & \frac{21875000h^2}{351562341} \\ \frac{594621789875043h}{3762708900050000} & -\frac{1919411007643420 16h}{7160447320494853125} & \frac{3794625470273293h}{43402593750000000 0} & -\frac{16072408962434637h}{73242154375000000000} \\ \frac{650015074642691961887h}{1574749451296972800000} & \frac{110283491311709113h}{458506372556700000} & -\frac{884523696838078255 93h}{282351744000000000000} & \frac{46481540204973773508 1h}{899999592960000000000 0} \\ \frac{48746093750000h}{130716507187737} & \frac{28208000000000000h}{61866264849075531} & \frac{3984269h}{37499841} & -\frac{156197h}{1406249364} \\ \frac{250000000000000h}{130716507187737} & -\frac{1280000000000000 00h}{61866264849075531} & \frac{89999788h}{37499841} & \frac{97187447h}{351562341} \end{bmatrix},$$

$$G_m^{[j]}=\begin{bmatrix} g_{n+\frac{579}{2500}} \\ g_{n+\frac{13493}{20000}} \\ g_{n+1} \\ g_{n+2} \end{bmatrix}$$

This can then be written explicitly as

$$
\begin{aligned}
\left(\varphi_{n+\frac{579}{2500}}-\phi_{n+\frac{579}{2500}}\right)&=\left(\varphi_n-\phi_n\right)+\tfrac{579}{2500}h\left(\varphi'_n-\phi'_n\right)+\tfrac{335241}{12500000}h^2\left(\varphi''_n-\phi''_n\right)+\tfrac{199681921435159082763}{1475796875000000000000000}h^3g_n\\
&+\tfrac{287179883899939001259}{3292370287543750000000000}h^3g_{n+\frac{579}{2500}}-\tfrac{149815028239099005693 6}{65264493806593713378906 25}h^3g_{n+\frac{13493}{20000}}\\
&+\tfrac{194091288755995335249}{2531817968750000000000000}h^3g_{n+1}-\tfrac{251906452279507952322 3}{128173770156250000000000000}h^3g_{n+2}\\
\left(\varphi_{n+\frac{13493}{20000}}-\phi_{n+\frac{13493}{20000}}\right)&=\left(\varphi_n-\phi_n\right)+\tfrac{13493}{20000}h\left(\varphi'_n-\phi'_n\right)+\tfrac{182061049}{800000000}h^2\left(\varphi''_n-\phi''_n\right)++\tfrac{2598297125889759926139385 9}{16601088000000000000000000000}h^3g_n\\
&+\tfrac{17411315486201411134678603969 57}{49972049254490603520000000000000}h^3g_{n+\frac{579}{2500}}+\tfrac{849140035860136382538191}{2567635686317520000000000000}h^3g_{n+\frac{13493}{20000}}\\
&+\tfrac{198151574984717543171381695 39}{537597720576000000000000000000000}h^3g_{n+1}-\tfrac{147684785276254666071183575 39}{1007999544115200000000000000000000}h^3g_{n+2}\\
\left(\varphi_{n+1}-\phi_{n+1}\right)&=\left(\varphi_n-\phi_n\right)+h\left(\varphi'_n-\phi'_n\right)+\tfrac{1}{2}h^2\left(\varphi''_n-\phi''_n\right)+\tfrac{146871661}{4374970320}h^3g_n+\tfrac{32845703125000}{305005183438053}h^3g_{n+\frac{579}{2500}}\\
&+\tfrac{11576000000000000}{433063853943528717}h^3g_{n+\frac{13493}{20000}}-\tfrac{3516367}{2624988870}h^3g_{n+1}+\tfrac{3126113}{196874910960}h^3g_{n+2}\\
\left(\varphi_{n+2}-\phi_{n+2}\right)&=\left(\varphi_n-\phi_n\right)+2h\left(\varphi'_n-\phi'_n\right)+2h^2\left(\varphi''_n-\phi''_n\right)+\tfrac{57809161}{820306935}h^3g_n+\tfrac{40037500000000}{53824444136127}h^3g_{n+\frac{579}{2500}}\\
&+\tfrac{48486400000000000}{433063853943528717}h^3g_{n+\frac{13493}{20000}}+\tfrac{30514444}{7705555}h^3g_{n+1}g_{n+1}+\tfrac{145312871}{12304681935}h^3g_{n+2}\\
\left(\varphi'_{n+\frac{579}{2500}}-\phi'_{n+\frac{579}{2500}}\right)&=\left(\varphi'_n-\phi'_n\right)+\tfrac{579}{2500}h\left(\varphi''_n-\phi''_n\right)+\tfrac{158597272523530029}{105414062500000000000}h^2g_n+\tfrac{26259788815091061}{1881354450025000000}h^2g_{n+\frac{579}{2500}}\\
&-\tfrac{4808499816880527424}{1491759858436427734375}h^2g_{n+\frac{13493}{20000}}+\tfrac{192903867854327017}{180844140625000000000}h^2g_{n+1}-\tfrac{2482984298382545709}{915526929687500000000 0}h^2g_{n+2}\\
\left(\varphi'_{n+\frac{13493}{20000}}-\phi'_{n+\frac{13493}{20000}}\right)&=\left(\varphi'_n-\phi'_n\right)+\tfrac{13493}{20000}h\left(\varphi''_n-\phi''_n\right)+\tfrac{4090459695927236804171}{8893440000000000000000}h^2g_n\\
&+\tfrac{849504962657530852893406 49}{5354148134409707520000000000}h^2g_{n+\frac{579}{2500}}+\tfrac{320309902250502823 03}{1222683660151200000000}h^2g_{n+\frac{13493}{20000}}\\
&-\tfrac{1060437283714988638053101}{3199986432000000000000000000}h^2g_{n+1}+\tfrac{644962671065705629639303}{1799999185920000000000000000}h^2g_{n+2}\\
\left(\varphi'_{n+1}-\phi'_{n+1}\right)&=\left(\varphi'_n-\phi'_n\right)+h\left(\varphi''_n-\phi''_n\right)+\tfrac{12187129}{187498728}h^2g_n+\tfrac{37456250000000}{130716507187737}h^2g_{n+\frac{579}{2500}}\\
&+\tfrac{3059200000000000}{20622088283025177}h^2g_{n+\frac{13493}{20000}}-\tfrac{53}{49999788}h^2g_{n+1}+\tfrac{312553}{2812498728}h^2g_{n+2}\\
\left(\varphi'_{n+2}-\phi'_{n+2}\right)&=\left(\varphi'_n-\phi'_n\right)+2h\left(\varphi''_n-\phi''_n\right)-\tfrac{2500106}{23437341}h^2g_n+\tfrac{168662500000000}{130716507187737}h^2g_{n+\frac{579}{2500}}\\
&-\tfrac{9881600000000000}{20622088283025177}h^2g_{n+\frac{13493}{20000}}+\tfrac{15416596}{12499947}h^2g_{n+1}+\tfrac{21875000}{351562341}h^2g_{n+2}\\
\left(\varphi''_{n+\frac{579}{2500}}-\phi''_{n+\frac{579}{2500}}\right)&=\left(\varphi''_n-\phi''_n\right)+\tfrac{774600153924897}{8433125000000000}hg_n+\tfrac{594621789875043}{3762708900050000}hg_{n+\frac{579}{2500}}-\tfrac{191941100764342016}{7160447320494853125}hg_{n+\frac{13493}{20000}}\\
&+\tfrac{3794625470273293}{434025937500000000}hg_{n+1}-\tfrac{16072408962434637}{732421543750000000 0}hg_{n+2}\\
\left(\varphi''_{n+\frac{13493}{20000}}-\phi''_{n+\frac{13493}{20000}}\right)&=\left(\varphi''_n-\phi''_n\right)+\tfrac{231937300020434917}{444672000000000000000}hg_n+\tfrac{650015074642691961887}{1574749451296972800000}hg_{n+\frac{579}{2500}}\\
&+\tfrac{110283491311709113}{458506372556700000}hg_{n+\frac{13493}{20000}}-\tfrac{884523696838078255 93}{28235174400000000000000}hg_{n+1}+\tfrac{464815402049737735081}{8999995929600000000000000}hg_{n+2}\\
\left(\varphi''_{n+1}-\phi''_{n+1}\right)&=\left(\varphi''_n-\phi''_n\right)+\tfrac{6093485}{93749364}hg_n+\tfrac{48746093750000}{130716507187737}hg_{n+\frac{579}{2500}}+\tfrac{28208000000000000}{61866264849075531}hg_{n+\frac{13493}{20000}}+\tfrac{3984269}{37499841}hg_{n+1}\\
&-\tfrac{156197}{1406249364}hg_{n+2}\\
\left(\varphi''_{n+2}-\phi''_{n+2}\right)&=\left(\varphi''_n-\phi''_n\right)-\tfrac{12187553}{23437341}hg_n+\tfrac{250000000000000}{130716507187737}hg_{n+\frac{579}{2500}}-\tfrac{128000000000000000}{61866264849075531}hg_{n+\frac{13493}{20000}}+\tfrac{89999788}{37499841}hg_{n+1}\\
&+\tfrac{97187447}{351562341}hg_{n+2}
\end{aligned}
\qquad (1.26)
$$

## III. Results and Discussion

### A. Analysis of Order and Error Constant of the New Two-Step Hybrid Optimized Volterra Integral Equation of the Second kind (1.26)

According to Chollom *et.al* (2007), let the linear difference operator 𝒷 associated with the new method (3.42) be defined by

$$L\left[\varphi(x);h\right]=\sum_{j=0}^{k}(\varphi-\phi)(x+jh)-h^{3}\left(\sum_{j=0}^{3}\mu_i x^{j}+\sum_{j=1}^{4}\psi_j e^{x^{j}}\right)$$

(1.27)

Where $\left(\varphi(x)-\phi(x)\right)$ is the exact solution satisfying equation (1.27). It can be expressed by Taylor's series expansion around the point $x_n$ to drive the statement

$$\ell\left[\left(\varphi(x)-\phi(x)\right):h\right]=\overline{c_o}\left(\varphi(x)-\phi(x)\right)+\overline{c_1}h\left(\varphi'(x)-\phi'(x)\right)+\overline{c_2}h^2\left(\varphi''(x)-\phi''(x)\right)$$
$$+\overline{c_3}h^3\left(\varphi'''(x)-\phi'''(x)\right)+\cdots+\overline{c_{p+3}}h^{p+3}\left(\varphi^{(p+3)}(x)-\phi^{(p+3)}(x)\right)+\cdots$$

(1.28)

Similarly,

The newly formulated Two-Step Hybrid Optimized Volterra Integral Equation of the second kind (1.26) is of order p if,

$$\ell\left[\left(\varphi(x)-\phi(x)\right):h\right]=o\left(h^{p+3}\right),\overline{c_o}=\overline{c_1}=\overline{c_2}=\overline{c_3}=\cdots=\overline{c_{p+2}}=0,\overline{c_{p+3}}\neq 0$$

Consequently, the primary local truncation error $x_n+k$ defined as

$$\overline{c_{p+3}}h^{p+3}\left(\varphi-\phi\right)^{p+3}\left(x_n\right)$$

Where

$$C_0=\sum_{j=0}^{k}\mu_j$$

$$C_1=\sum_{j=1}^{k}j\mu_j-\sum_{j=0}^{k}\psi_j$$

$$C_2=\frac{1}{2}\sum_{j=1}^{k}j^2\mu_j-\sum_{j=0}^{k}j\psi_j$$

. (1.29)

.

.

$$C_q=\frac{1}{q!}\sum_{j=1}^{k}j^q\mu_j-\frac{1}{(q-1)!}\sum_{j=0}^{k}j^{q-1}\psi_j-\frac{1}{(q-2)!}\sum_{j=0}^{k}j^{q-2}\varpi_j \quad , \quad q=3\,,\,4\,,\,5\,,\,...$$

 

Hence, the new methods (1.26) has order

$c_{p+2} = 7$ , in which its order is $p = (5,5,5,5,5,5,5,5,5,5)^T$ , with error constant

$$c_8 = \left[ \frac{4680111\,685481992\,064528991}{27343\,750000000\,000000000\,000000000}, \frac{8323045\,544727457\,197244045\,509357389}{5160\,960000000\,000000000\,000000000\,000000000}, \frac{907747}{1008\,000000000}, -\frac{34957753}{157\,500000000}, -\frac{62075\,807249320\,721749707}{27\,343750000\,000000000\,000000000}, \frac{1990015711\,021791456\,665206477}{3225600\,000000000\,000000000\,000000000}, -\frac{2006131}{630\,000000000}, -\frac{143612753}{157\,500000000}, \frac{27\,453486944\,178846471}{1562500\,000000000\,000000000}, -\frac{1895764\,605078798\,975121999}{76\,800000000\,000000000\,000000000}, \frac{74821}{24\,000000000}, -\frac{21731}{9000000} \right]$$

**B. Zero Stability of the New Two-Step Hybrid Optimized Volterra Integral Equation of the Second Kind (1.26)**

The novel two-step hybrid optimizes the Volterra Integral Equation of the second Kind (1.26) for k=2, utilizing two off-grid collocation points articulated in the following manner.

$$\rho(z) = \det\left[ \sum_{j=0}^{k} A^{(i)} z^{k-i} \right] = 0$$

(1.30)

Gives

$$\rho(z) = \left| z \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} - \begin{bmatrix} 0&0&0&1&0&0&0&\frac{597}{2500}&0&0&0&\frac{356409}{12500000} \\ 0&0&0&1&0&0&0&\frac{13493}{20000}&0&0&0&\frac{182061049}{800000000} \\ 0&0&0&1&0&0&0&1&0&0&0&\frac{1}{2} \\ 0&0&0&1&0&0&0&2&0&0&0&2 \\ 0&0&0&0&0&0&0&1&0&0&0&\frac{597}{2500} \\ 0&0&0&0&0&0&0&1&0&0&0&\frac{13493}{20000} \\ 0&0&0&0&0&0&0&1&0&0&0&1 \\ 0&0&0&0&0&0&0&1&0&0&0&2 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \right|$$

Finding its determinant and solving the characteristic equation yields

$$= (z-1)^3 z^9$$

$$\rho(\text{z}) = (z-1)^3 z^9 = 0, \quad z = 0,0,0,0,0,0,0,0,0,1,1,1$$

since

$$|z_i| = |0,0,0,0,0,0,0,0,0,1,1,1| \le 1$$

The approach exhibits zero-stability.

**C. Convergence of the New Two-Step Hybrid Optimized Volterra Integral Equation of the second kind (1.26)**

**Theorem 4.1** (Henrici, 1962). Zero stability and consistency are necessary requirements for the convergence of a linear multistep algorithm. Since our New Two-Step Hybrid Optimized Volterra Integral Equation of the second kind (1.26) is consistent and zero-stable, by Henrici (1962). The New Two-Step Hybrid Optimized Volterra Integral Equation of the second kind (1.26) exhibits convergence.

**D. Region of Complete Stability of Novel Two-Phase Hybrid Optimized Volterra Integral Equation of the second sort (1.26)**

To delineate the region of absolute stability for the New Two-Step Hybrid Optimized Volterra Integral Equation of the second sort (1.26), the methodologies were articulated as a generic linear method represented as

$$A^{(0)}(\varphi-\phi)_m^{(i)} = \sum_{i=0}^{2} e_i(\varphi-\phi)_n^{(i)} - h^3\left[b_i g(\varphi_n-\phi_n) + d_i g(\varphi_m-\phi_m)^w\right] \qquad (1.31)$$

$$(\varphi-\phi)_m^{(i)} = \left[(\varphi-\phi)_{n+\frac{597}{2500}}^{(i)} \quad (\varphi-\phi)_{n+\frac{13493}{20000}}^{(i)} \quad (\varphi-\phi)_{n+1}^{(i)} \quad (\varphi-\phi)_{n+2}^{(i)}\right]^T$$

$$(\varphi-\phi)_n^{(i)} = \left[(\varphi-\phi)_{n-\frac{597}{2500}}^{(i)} \quad (\varphi-\phi)_{n-\frac{13493}{20000}}^{(i)} \quad (\varphi-\phi)_{n-1}^{(i)} \quad (\varphi-\phi)_{n}^{(i)}\right]^T$$

Where

$$G(\varphi_m-\phi_m) = \left[g_{n+\frac{597}{2500}}^{(i)} \quad g_{n+\frac{13493}{20000}}^{(i)} \quad g_{n+1}^{(i)} \quad g_{n+2}^{(i)}\right]^T$$

$$\text{g}(\varphi_n-\phi_n) = \left[g_{n-\frac{597}{2500}}^{(i)} \quad g_{n-\frac{13493}{20000}}^{(i)} \quad g_{n-1}^{(i)} \quad g_{n}^{(i)}\right]^T$$

Where

$A^{(0)} = 4 \times 4$ Identity matrix

When $i = 2$

$$e_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad b_0 = \begin{bmatrix} 0 & 0 & 0 & \frac{792932\,838876567}{8433125\,000000000} \\ 0 & 0 & 0 & \frac{264607749\,195427957}{4584960000\,000000000} \\ 0 & 0 & 0 & \frac{6767855}{96663852} \\ 0 & 0 & 0 & -\frac{11944679}{24165963} \end{bmatrix},$$

$$d_0 = \begin{bmatrix} \frac{602980\,713552453}{3651947\,487650000} & -\frac{209170973\,213305088}{7044082980\,787003125} & \frac{4125708\,015479389}{429959062\,500000000} & -\frac{17440710\,462939501}{72\,943950625\,000000000} \\ \frac{650\,015074642\,691961887}{1575\,909002633\,011200000} & \frac{105164039\,772064333}{451055191\,239900000} & -\frac{1386\,058302342\,635723801}{47550\,032640000\,000000000} & \frac{24\,584238359\,491883753}{52725\,603840000\,000000000} \\ \frac{48746\,093750000}{130812\,759007623} & \frac{27344000\,000000000}{60860876\,953999707} & \frac{4020017}{37148463} & -\frac{219071}{1400523852} \\ \frac{250000\,000000000}{130812\,759007623} & -\frac{128000000\,000000000}{60860876\,953999707} & \frac{89531284}{37148463} & \frac{96710321}{350130963} \end{bmatrix}$$

(1.3)

Subsequently, calculations were performed utilizing scientific workplace software to derive the stability polynomial for technique (1.26) utilizing (1.30).

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} w - \begin{bmatrix} 0 & 0 & 0 & \frac{792932\,838876567}{8433125\,000000000} \\ 0 & 0 & 0 & \frac{264607749\,195427957}{4584960000\,000000000} \\ 0 & 0 & 0 & \frac{6767855}{96663852} \\ 0 & 0 & 0 & -\frac{11944679}{24165963} \end{bmatrix} h$$

$$-h \begin{bmatrix} \frac{602980\,713552453}{3651947\,487650000} & -\frac{209170973\,213305088}{7044082980\,787003125} & \frac{4125708\,015479389}{429959062\,500000000} & -\frac{17440710\,462939501}{72\,943950625\,000000000} \\ \frac{650\,015074642\,691961887}{1575\,909002633\,011200000} & \frac{105164039\,772064333}{451055191\,239900000} & -\frac{1386\,058302342\,635723801}{47550\,032640000\,000000000} & \frac{24\,584238359\,491883753}{52725\,603840000\,000000000} \\ \frac{48746\,093750000}{130812\,759007623} & \frac{27344000\,000000000}{60860876\,953999707} & \frac{4020017}{37148463} & -\frac{219071}{1400523852} \\ \frac{250000\,000000000}{130812\,759007623} & -\frac{128000000\,000000000}{60860876\,953999707} & \frac{89531284}{37148463} & \frac{96710321}{350130963} \end{bmatrix} w$$

(1.33)

Simplifying and finding its determinant gives

$$h(w) = -h^4\left(\frac{116710321}{3000000000}w^3 - \frac{2685107}{1000000000}w^4\right) - h^3\left(\frac{115510963}{3000000000}w^4 + \frac{219595321}{1000000000}w^3\right) + h^2\left(\frac{245072821}{1000000000}w^4 - \frac{679692821}{1000000000}w^3\right)$$
$$-h\left(\frac{78269}{100000}w^4 + \frac{121731}{100000}w^3\right) + w^4 - w^3$$

The absolute stability region of method (1.26) is displayed using Mat Lab software.

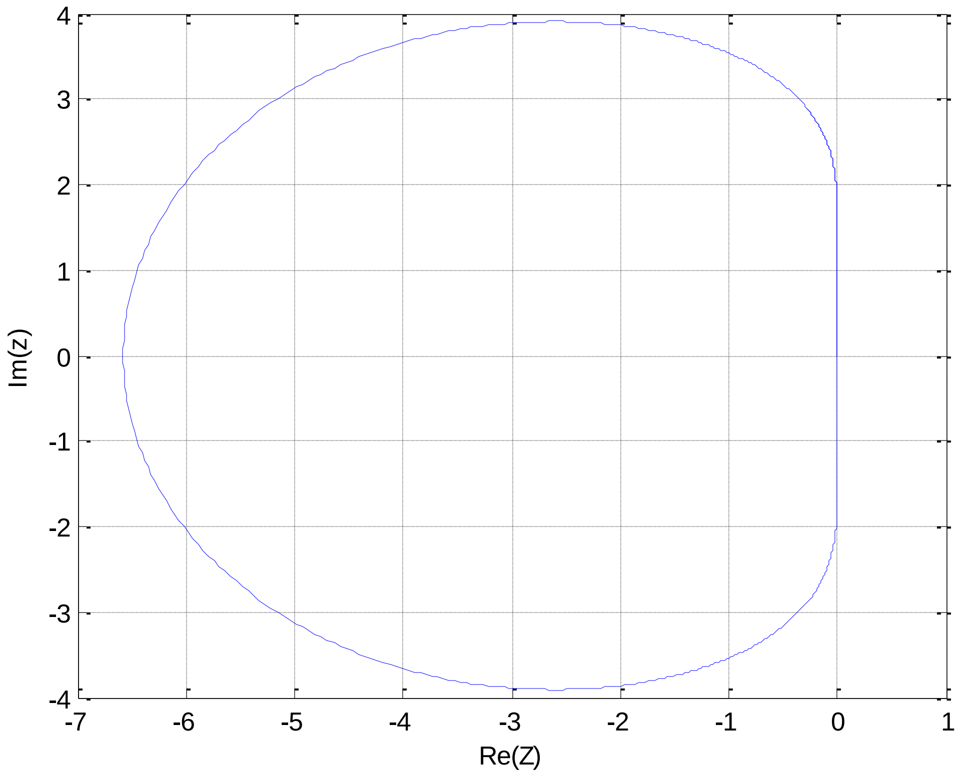


**Figure1.**Region of Absolute Stability (RAS) for the fifth-order technique (1.26)

**Table1 Overview of the examination of the methodologies**

| **Method** | **Order** | **Consistency** | **Zero stability** | **Error Constant** |
|---|---|---|---|---|
| CASE1 (1.26) | P = 5 | Consistent | Zero stable | $C_{10}=\left[\frac{457}{34681651200},\right]$ |

## E. Implementation of the Developed Numerical Method

The implementation strategy for the third-order initial value problems was reconstructed as Volterra integral equations of the second class and resolved utilizing optimized hybrid numerical methods. The integration of power series and exponential fitting markedly improved stability and precision. Symbolic and numerical implementations validated the theoretical properties, while numerical experiments exhibited enhanced performance compared to previous approaches.

**F. Numerical Example**

1. Consider the third order linear problem

$$y''' = -\exp(x)$$

$$y(0)=1, y'(0)=-1, y''(0)=3$$ Source: Omole, E.O. et al. (2024)

$$Exact\ Solution;: y(x) = 2 + 2x^2 - e^x,\ h = \frac{1}{10}$$

The following notations are used in the tables

$X$ Points of evaluation

$Yex$ Exact Solution

2S2OHM Two step, two optimized hybrid point method

ERR Absolute Error

The calculated outcomes for the problems utilizing the two proposed case approaches are displayed in tables.

**Table2.Presenting the exact solution and calculated outcomes derived from the proposed approaches for Problem 1.**

| X | Yex | 2S2OHM |
|---|---|---|
| 0.1 | 0.91482908192435237520 | 0.91482908192435237885 |
| 0.2 | 0.85859724183983016610 | 0.85859724183983013061 |
| 0.3 | 0.83014119242399689600 | 0.83014119242399683510 |
| 0.4 | 0.82817530235872968220 | 0.82817530235872965176 |
| 0.5 | 0.85127872929987185320 | 0.85127872929987181696 |
| 0.6 | 0.89788119960949102510 | 0.89788119960949104595 |
| 0.7 | 0.96624729252952347840 | 0.96624729192265255995 |
| 0.8 | 1.05445907150753239540 | 1.05445907150753295843 |
| 0.9 | 1.16039688884305033620 | 1.16039688884305097255 |
| 1.0 | 1.28171817154095476460 | 1.28171817154095430396 |

**Table 3 Comparing the absolute errors in the new methods with error from Omole *et al* (2024).**

| Error in 2S2OHM | Error in Omole et al. (2024) |
|---|---|
| 3.65000e-18 | 8.1100e-17 |
| 3.54900e-17 | 1.4010e-16 |
| 6.09000e-17 | 2.0410e-16 |
| 3.04400e-17 | 2.7010e-16 |
| 3.62400e-17 | 3.4810e-16 |
| 1.66000e-17 | 4.4310e-16 |
| 2.08500e-16 | 5.3510e-16 |
| 5.63400e-16 | 6.4410e-16 |
| 6.36400e-16 | 7.6410e-16 |
| 4.60600e-16 | 8.8410e-16 |

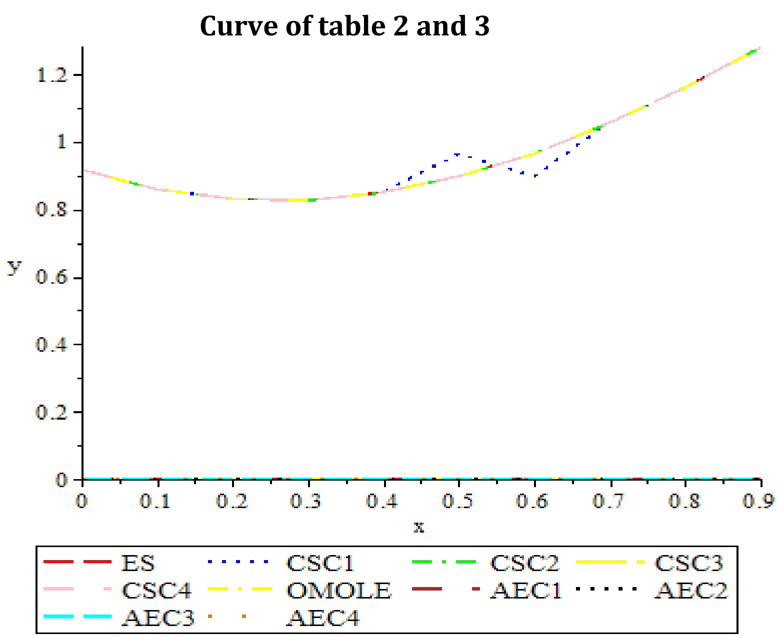


Figure 2: Graphical illustration of the absolute errors in the new methods with error from Omole *et al*. (2024)

## G. Discussion of the Result

This study developed a new Class of two-step with two optimized hybrid methods for third derivative methods using the combination of power series and an exponential fitted for deriving Volterra integral schemes of the second kind. Figure 1 illustrates the stability region plot of the two-step optimized hybrid approaches. The optimized hybrid Volterra methods clearly show superior stability coverage, particularly in the left-half complex plane. The newly optimized hybrid Volterra methods were solved and tested on one problem using third order stiff linear problems. The problems were taken from the works of Omole et al (2024). Our results converged faster and gave better approximations which show a higher superiority over existing methods as cited and their exact solutions.

## H. Conclusion

There is a gap in methods that integrate Volterra formulations with hybrid multistep schemes and parameter optimization, particularly for stiff systems. This work addresses: How can a hybrid Volterra integral equation of the second kind be optimized to solve third-order IVPs with enhanced accuracy, reduced computational overhead and propagation errors.

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