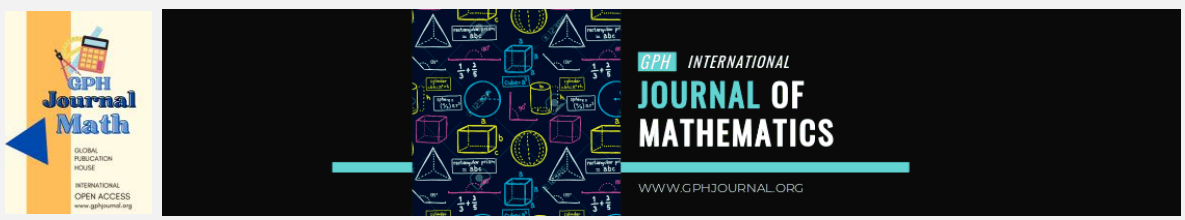
e-ISSN 3050-9645
p-ISSN 2795-3274



10.5281/zenodo.16364036 Vol. 8 Issue 05 May - 2025 Manuscript ID: #2018

Fractional mathematical model for the dynamics of pneumonia transmission with control using fixed point theory

[1,2]Lemmy Michael Emmanuel, [1,2]David Omale, [1,2]William Atokolo, [1,2]Jeremiah Amos, [1,2]Emmanuel Abah, [1,2]Anibe Alexander Ojonimi, [1,2]Thomas Onoja, [1,2]Godwin Onuche Acheneje, [1,2]Bolarinwa Bolaji.

[1]Department of Mathematical Sciences, Prince Abubakar Audu University, Anyigba, Nigeria
[2]Laboratory of Mathematical Epidemiology, Prince Abubakar Audu University, Anyigba.,
***Corresponding Author:** williamsatokolo@gmail.com

Abstract

This study looks at key epidemiological details of pneumonia by using a mathematical model that includes fixed point theory and fractional-order calculations to see how treatment and vaccination affect transmission. Using fixed point theory for numerical simulations, it is easy to show the relationship between pneumonia dynamics and the different values and parameters in fractional-order models. Through more analyses, it has been shown that both rising contact and weaker treatment would result in an increase in pneumonia cases. Also, the study shows that increasing the numbers of vaccinated and treated individuals can fight and reduce the occurrence of the disease among humans.

Keywords:

Pneumonia, Fractional, simulation, Fixed point theory.

How to cite: Emmanuel, L., Omale, D., Atokolo, W., Amos, J., Abah, E., Ojonimi, A., Onoja, T., Acheneje, G., & Bolaji, B. (2025). Fractional mathematical model for the dynamics of pneumonia transmission with control using fixed point theory. *GPH-International Journal of Mathematics*, 8(5), 55-86. https://doi.org/10.5281/zenodo.16364036

1.0 Introduction

For the past decades, scientists have found it very important to model biological, physical, and chemical phenomena and processes by using mathematical ideas. Our research shows that mathematical models come in handy for exploring different issues that keep evolving in the physical and life sciences. He envisioned the idea in 1776. Rather, McKendrick developed the SIR model in 1927, which divides the whole population into people who are susceptible, infectious, and recovered. At that point, mathematical modeling started

addressing infectious diseases. Mathematics is applied to the study of biology these days, allowing for better understanding of infectious diseases and new ways to restrict their spread. Experts have created many mathematical models to get a better grasp of infectious diseases and discover treatments for them [1–4].

These studies make a big impact on several scientific areas. To understand how pneumonia develops and give suggestions for better treatment, the researchers used fractional-order derivatives.

Infectious diseases are clear medical conditions, and they usually have a major effect on people around the world. The change in neural cells happens when a person is affected by bacterial, viral, fungal, parasitic, or by the protein disease agents called prions. Many people suffer from tuberculosis and pneumonia caused by bacteria, HIV, and influenza caused by a virus [6, 7].

Pneumonia occurs when bacteria, viruses, fungi, or parasites attack the lungs or their inner parts which are called alveoli [8–9]. Usually, those who get pneumonia are seniors, babies, and people who suffer from other medical conditions or have weak immune systems. More often than not, pneumonia is caused by Streptococcus pneumoniae, which is known as pneumococcus [10-11]. Treatments and vaccination efforts are used as the main ways to deal with pneumonia infection [8]. Tilahun et al. [12] introduced and examined a new model of pneumonia transmission in a group with various sizes, and recommended control options and best ways to address the disease. The researchers found that using prevention and treatment measures is the best way to tackle the problem of pneumonia, since it is the most cost-effective choice. Ndelwa et al. [13] made a mathematical model and studied various strategies for tackling pneumonia infection. They found that screening and treating the infection at once is the best way to remove the pneumonia epidemic from the community. Maximum control over the HIV/AIDS-pneumonia co-infection was the main aim of Nthiiri et al.'s project, as they analyzed the greatest level of protection possible for both conditions [14]. They did not concentrate on preventing the spread of both infections combined; their strategy aimed to prevent infection by just one. Much of the included studies neglected to address treatment for the sub-models or for the people with both infections. It turned out that when people are well protected, there are fewer HIV/AIDS and pneumonia cases. [15] built a fractional Caputo derivative compartmental model to explore soil-transmitted helminth infection dynamics and solved it using the Laplace-transform approach. The researchers showed that the exact solution is reached by the infinite series, and a fractional order model is more flexible than a classical model. Atokolo et. al. [16] developed a fractional order sterile insect technology (SIT) model to work against Zika virus using the Laplace–Adomian decomposition method (LADM) to find an analytical solution. It was shown that the flexibility of the fractional model lets you change the response by varying its order in a fraction. They proved that LADM is useful in resolving SIT models, Sweden shows a new and practical way in the field. The results of their work showed that limiting the number of contacts and enhancing how the disease is treated worked well to manage it as well as improved flexibility of fractional models compared to the traditional models. James et al. [18], came up with a fractional-order HIV/AIDS transmission model that uses the Adams-

Bashforth-Moulton numerical method. According to them, dealing with the disease is most successful when people interact less and doctors provide new treatments, as fractional models are shown to be much more flexible than traditional ones. Abah et al. [19] presented a fractional-order model for Diphtheria transmission along with using the Adams-Bashforth-Moulton method. It was shown that fewer contacts and improved successful treatments with fractional methods limited the spread of the disease compared to traditional ones. Atokolo et al. [20] looked into Lassa fever using a fractional-order mathematical model, considering the use of a power-law fractional derivative to check how vaccination and treatment influence the way the disease spreads.

2.0 Model flow diagram

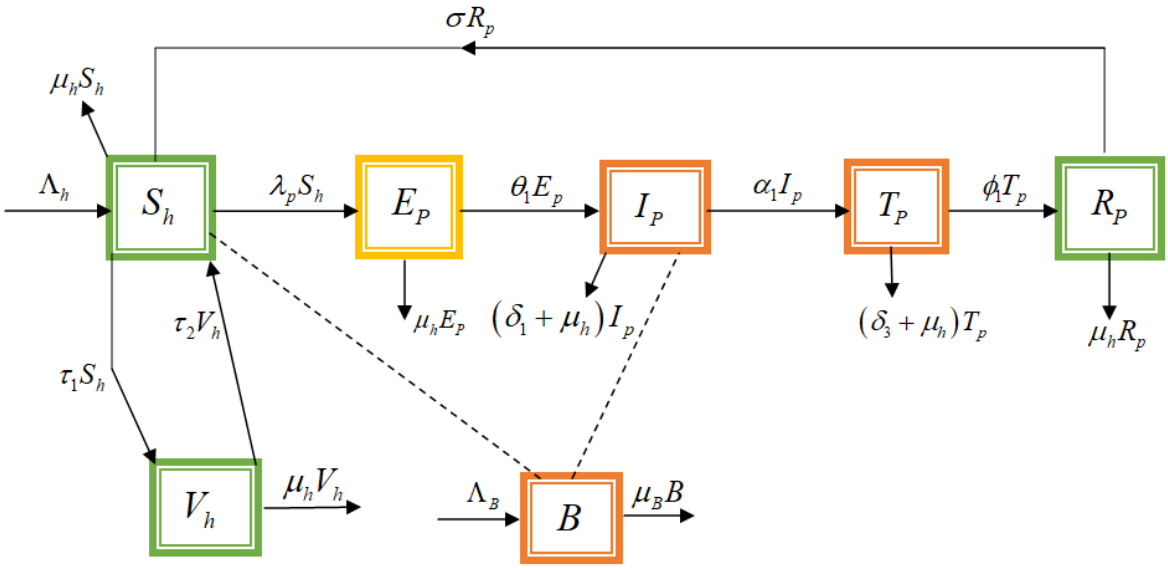


Fig.1: Pneumonia model Flow Diagram

2.1 Model equation

$$\frac{dS_h}{dt} = \Lambda_h + \tau_2 V_h + \sigma R_p - \lambda_p S_h - (\tau_1 + \mu_h) S_h,$$

$$\frac{dE_P}{dt} = \lambda_p S_h - (\theta_1 + \mu_h) E_P,$$

$$\frac{dI_P}{dt} = \theta_1 E_P - (\alpha_1 + \delta_1 + \mu_h) I_P,$$

$$\frac{dT_P}{dt} = \alpha_1 I_P - (\phi_1 + \delta_3 + \mu_h) T_P, \qquad \textbf{(1)}$$

$$\frac{dV_h}{dt} = \tau_1 S_h - (\tau_2 + \mu_h) V_h,$$

$$\frac{dR_p}{dt} = \phi_1 T_p - (\sigma + \mu_h) R_p,$$

$$\frac{dB}{dt} = \Lambda_B - \mu_B B.$$

Where $\lambda_p = \frac{(\beta_1 I_p + \beta_2 T_p + \beta_3 B)}{N_h}$.

2.2 Model analysis

2.3 Fractional order model

$$^{LC}_{0}D_t^{\alpha} S_h = \Lambda_h + \tau_2 V_h + \sigma R_p - \lambda_p S_h - (\tau_1 + \mu_h) S_h,$$

$$^{LC}_{0}D_t^{\alpha} E_P = \lambda_p S_h - (\theta_1 + \mu_h) E_P,$$

$$^{LC}_{0}D_t^{\alpha} I_P = \theta_1 E_P - (\alpha_1 + \delta_1 + \mu_h) I_P,$$

$$^{LC}_{0}D_t^{\alpha} T_P = \alpha_1 I_P - (\phi_1 + \delta_3 + \mu_h) T_P, \qquad \textbf{(2)}$$

$$^{LC}_{0}D_t^{\alpha} V_h = \tau_1 S_h - (\tau_2 + \mu_h) V_h,$$

$$^{LC}_{0}D_t^{\alpha} R_p = \phi_1 T_p - (\sigma + \mu_h) R_p,$$

$$^{LC}_{0}D_t^{\alpha} B = \Lambda_B - \mu_B B.$$

Subject to the initial condition

$$S_h(0) \geq 0, \mathrm{E}_p(0) \geq 0, \mathrm{I}_p(0) \geq 0, \mathrm{T}_p(0) \geq 0, \mathrm{V}_h(0) \geq 0, \mathrm{R}_p(0) \geq 0, \mathrm{B}(0) \geq 0.$$

Basic Definition

Definition 1: According to Caputo, the derivative of a function of order α that is not an integer is defined as shown below [23]:

$$^{C}D_{\theta}^{n}\psi(V,\theta)=\frac{1}{\Gamma(n-\phi)}\int_{0}^{\theta}(\theta-\alpha)^{n-\phi-1}\psi^{(n)}(V,\alpha)d\alpha \qquad (3)$$

$n-1<\rho\leq n,\ t>0.$

Definition 2: Derovich et al [23] describe the non-integer Riemann integral as shown below:

$$R_{\theta}^{\phi}\psi(V,\theta)=\frac{1}{\Gamma(\theta)}\int_{0}^{\theta}(\theta-\alpha)^{\phi-1}\psi(V,\theta)d\theta \qquad (4)$$

Definition 3: The Laplace transform (LT) of $\psi(V,\theta)$ is presented by Syafruddin and Nooarani [23] as follows;

$$\psi(\omega,\mathrm{S})=L_{\theta}\left[\psi(V,\theta)\right]$$

$$\psi(\omega,\mathrm{S})=\int_{0}^{\infty}e^{-S\theta}\psi(V,\gamma)d\theta,\ S>\phi. \qquad (5)$$

The following is the definition for the inverse Laplace transform

$$\psi(V,\theta)=L_{\theta}^{-1}\left[\psi(V,\mathrm{S})\right]$$

$$\psi(V,\theta)=\int_{b-i\infty}^{b+i\infty}e^{S\theta}\psi(V,\mathrm{S})dS,\ b=\mathrm{Re}(s)>b_{0} \qquad (6)$$

Lemma 1: for $m-1<K\leq m,\ \varepsilon>-1,\ \phi\geq 0.$ we have;

1. $D_{\theta}^{K}\theta^{\tau}=\frac{\Gamma(k+1)}{\Gamma(\tau-k-1)}\theta^{\tau-k}$
2. $D_{\theta}^{K}\theta=0.$
3. $D_{\theta}^{k}R_{\theta}^{k}\psi(V,\theta)=\psi(V,\theta)$
4. $D_{\theta}^{k}R_{\theta}^{k}\psi(V,\theta)=\psi(V,\theta)-\sum_{i=0}^{m-1}\partial^{i}\psi(V,)\frac{\theta^{i}}{i!}.$

Definition 4: Let $X\in Q'(a,b),\ b>a,\ \xi\in(0,1),$ then the given Caputo-Fabrizio fractional derivatives (CFFD) is $^{CF}_{0}D_{t}^{\xi}X(t)=\frac{k(\xi)}{1-\xi}\int_{\rho}^{t}X'(n)\exp\left[-\frac{t-n}{1-n}\right]\mathrm{dn},$ (7)

Where $k(\xi)$ in equation (7) satisfies $k(1)=k(0)=1$ in addition, if X is not in $Q'(a,b)$, then the equation becomes;

$$ {}^{CF}_{0}D_t^{\xi}X(t)=\frac{k(\xi)}{1-\xi}\int_{\rho}^{t}X(t)-X(\xi)\exp\left[-\frac{t-\xi}{1-\xi}\right]d\xi. $$

Definition 5: Let $\xi\in(0,1)$, and the integral of the function X to the fractional order ξ is

$$ {}^{CF}_{0}I_t^{\xi}X(t)=\frac{1-\xi}{k(\xi)}\int_{\rho}^{t}X(t)+\frac{\xi}{k(\xi)}\int_{0}^{t}X(n)dn. $$

Lemma 2: The problem that occurs with CFFD is that $\begin{cases} {}^{CF}_{0}D_t^{\xi}X(t)=Z(t),\ 0<\xi\le 1, \\ X(0)=X_0 \end{cases}$ where x is real constant and m other words, it is corresponding to the integral $X(t)=X_0+\frac{1-\xi}{k(\xi)}X(t)+\frac{\xi}{k(\xi)}\int_{0}^{t}X(n)dn.$

Definition 6:[23,24] The Caputo-Fabrizio fractional derivative of Laplace transform is ${}^{CF}_{0}D_t^{\xi}$, $\xi\in(0,1]$ of $M(t)$ is given as;

$$ L\left[{}^{CF}_{0}I_t^{\xi}N(t)\right]=\frac{SL\left[N(t)\right]-N(0)}{S+\xi(1-S)}. $$

Disease free Equilibrium point

At the disease-free equilibrium, people experience no signs of illness.

$$ \left(S_h,E_P^0,\mathrm{I}_P^0,\mathrm{T}_P^0,\mathrm{V}_P^0,\mathrm{R}_P^0,B^0\right)=\left(\frac{\Lambda_h(\tau_2+\mu_h)}{\mu_h(\tau_2+\tau_1+\mu_h)},0,0,0,\frac{\tau_1\Lambda_h}{\mu_h(\tau_2+\tau_1+\mu_h)},0,\frac{\Lambda_b}{\mu_b}\right). $$

2.4 Basic Reproduction Number

The complete number of infections going from one infected person to non-infected people is what the Basic Reproduction number means. To find the Basic Reproduction number, next generation method $R_0^P=\rho FV^{-1}$ is used and you have to determine the largest value ρ of the Eigen vector from F that is a non-negative matrix along with V.

$$F=\begin{pmatrix} 0 & \dfrac{\beta_1(\tau_2+\mu_h)}{\tau_2+\tau_1+\mu_h} & \dfrac{\beta_2(\tau_2+\mu_h)}{\tau_2+\tau_1+\mu_h} & \dfrac{\beta_3(\tau_2+\mu_h)}{\tau_2+\tau_1+\mu_h} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } V=\begin{pmatrix} P_2 & 0 & 0 & 0 \\ -\theta_1 & P_3 & 0 & 0 \\ 0 & -\alpha_1 & P_4 & 0 \\ 0 & 0 & 0 & P_7 \end{pmatrix}$$

$$R_0^P=\frac{\theta_1\left(P_4\beta_1\mu_h+P_4\beta_1\tau_2+\alpha_1\beta_2\mu_h+\alpha_1\beta_2\tau_2\right)}{P_2P_3P_4\left(\tau_2+\tau_1+\mu_h\right)}.$$

This is the dominant Eigen value.

2.5 Endemic Equilibrium point

Pneumonia remains constant among humans in the condition known as Endemic Equilibrium.

$$S_h^*=-\frac{\Lambda_hP_2P_3P_4P_5P_6}{\left(\left(-\lambda_P-P_1\right)P_5+\tau_1\tau_2\right)P_3P_4P_6P_2+\sigma P_5\alpha_1\lambda_P\phi_1\theta_1},$$

$$E_P^*=-\frac{\Lambda_hP_3P_4P_5P_6\lambda_P}{\left(\left(-\lambda_P-P_1\right)P_5+\tau_1\tau_2\right)P_3P_4P_6P_2+\sigma P_5\alpha_1\lambda_P\phi_1\theta_1},$$

$$I_P^*=-\frac{\Lambda_hP_4P_5P_6\lambda_P\theta_1}{\left(\left(-\lambda_P-P_1\right)P_5+\tau_1\tau_2\right)P_3P_4P_6P_2+\sigma P_5\alpha_1\lambda_P\phi_1\theta_1},$$

$$T_P^*=-\frac{\Lambda_hP_5P_6\lambda_P\theta_1\alpha_1}{\left(\left(-\lambda_P-P_1\right)P_5+\tau_1\tau_2\right)P_3P_4P_6P_2+\sigma P_5\alpha_1\lambda_P\phi_1\theta_1},$$

$$V_P^*=-\frac{\Lambda_hP_2P_3P_4P_6\tau_1}{\left(\left(-\lambda_P-P_1\right)P_5+\tau_1\tau_2\right)P_3P_4P_6P_2+\sigma P_5\alpha_1\lambda_P\phi_1\theta_1},$$

$$R_P^*=-\frac{P_5\alpha_1\lambda_P\phi_1\theta_1\Lambda_h}{\sigma P_5\alpha_1\lambda_P\phi_1\theta_1-P_1P_2P_3P_4P_5P_6-P_2P_3P_4P_5P_6\lambda_P+P_2P_3P_4P_6\tau_1\tau_2},$$

$$B^*=\frac{\Lambda_b}{\mu_b}.$$

Substituting into the force of infection $\lambda_p=\dfrac{\left(\beta_1I_p+\beta_2T_p+\beta_3B\right)}{N_h}$ we have:

$$P\left(\lambda_h^*\right)=M_1\lambda_h^*+M_2$$

$$K_1 = \sigma\Lambda_b P_5\alpha_1\phi_1\theta_1 - \Lambda_b P_2P_3P_4P_5P_6 - \Lambda_h P_3P_4P_5P_6\mu_b - \Lambda_h P_4P_5P_6\mu_b\theta_1 - \Lambda_h P_5P_6\alpha_1\mu_b\theta_1 - \Lambda_h P_5\alpha_1\mu_b\phi_1\theta_1,$$

$$K_2 = P_1P_5\left(1 - \frac{\theta_1\left(P_4\beta_1\mu_h + P_4\beta_1\tau_2 + \alpha_1\beta_2\mu_h + \alpha_1\beta_2\tau_2\right)}{P_2P_3P_4\left(\tau_2 + \tau_1 + \mu_h\right)}\right),$$

$$K_2 = P_1P_5\left(1 - R_0^P\right).$$

If $R_0^P > 1$It means that model (1) only possesses a special positive equilibrium point for endemic states.

2.6 Sensitivity Analysis

It helps discover which population disease transmission parameters bring good results. It can be determine using: $S_x^{R_0} = \left(\frac{\partial R_0}{\partial x}\right)\left(\frac{x}{R_0}\right)$.

$$S_{\theta_1}^{R_0^P} = \frac{\mu_h}{\theta_1 + \mu_h} = 0.0401,$$

$$S_{\phi_1}^{R_0^P} = -\frac{\alpha_1\beta_2\left(\mu_h + \tau_2\right)\phi_1}{\left(\beta_1\left(\tau_2 + 1\right)\mu_h{}^2 + \left(\left(\tau_2 + 1\right)\left(\phi_1 + \delta_3\right)\beta_1 + \alpha_1\beta_2\right)\mu_h + \beta_2\alpha_1\tau_2\right)\left(\phi_1 + \delta_3 + \mu_h\right)} = -0.9786,$$

$$S_{\alpha_1}^{R_0^P} = -\frac{\alpha_1\left(\left(\beta_1\left(\tau_2 + 1\right) - \beta_2\right)\mu_h{}^2 + \left(\left(\tau_2 + 1\right)\left(\phi_1 + \delta_3\right)\beta_1 - \beta_2\left(\tau_2 + \delta_1\right)\right)\mu_h - \tau_2\beta_2\delta_1\right)}{\left(\beta_1\left(\tau_2 + 1\right)\mu_h{}^2 + \left(\left(\tau_2 + 1\right)\left(\phi_1 + \delta_3\right)\beta_1 + \alpha_1\beta_2\right)\mu_h + \beta_2\alpha_1\tau_2\right)\left(\alpha_1 + \delta_1 + \mu_h\right)} = -0.92119,$$

$$S_{\tau_2}^{R_0^P} = \frac{\tau_2\left(\mu_h{}^3\beta_1 + \beta_1\left(\phi_1 + \tau_1 + \delta_3 - 1\right)\mu_h{}^2 + \beta_1\left(\tau_1 - 1\right)\left(\phi_1 + \delta_3\right)\mu_h + \tau_1\beta_2\alpha_1\right)}{\left(\tau_2 + \tau_1 + \mu_h\right)\left(\beta_1\left(\tau_2 + 1\right)\mu_h{}^2 + \left(\left(\tau_2 + 1\right)\left(\phi_1 + \delta_3\right)\beta_1 + \alpha_1\beta_2\right)\mu_h + \beta_2\alpha_1\tau_2\right)} = 0.5785,$$

$$S_{\tau_1}^{R_0^P} = -\frac{\tau_1}{\tau_2 + \tau_1 + \mu_h} = -0.59196,$$

$$S_{\beta_1}^{R_0^P} = \frac{\left(\left(\phi_1 + \delta_3 + \mu_h\right)\mu_h + \left(\phi_1 + \delta_3 + \mu_h\right)\tau_2\mu_h\right)\beta_1}{\left(\phi_1 + \delta_3 + \mu_h\right)\beta_1\mu_h + \left(\phi_1 + \delta_3 + \mu_h\right)\beta_1\tau_2\mu_h + \beta_2\alpha_1\mu_h + \beta_2\alpha_1\tau_2} = 0.01783,$$

$$S_{\beta_2}^{R_0^P} = \frac{\left(\alpha_1\mu_h + \alpha_1\tau_2\right)\beta_2}{\left(\phi_1 + \delta_3 + \mu_h\right)\beta_1\mu_h + \left(\phi_1 + \delta_3 + \mu_h\right)\beta_1\tau_2\mu_h + \beta_2\alpha_1\mu_h + \beta_2\alpha_1\tau_2} = 0.9822,$$

$$S_{\delta_1}^{R_0^P} = -\frac{\delta_1}{\alpha_1 + \delta_1 + \mu_h} = -0.10247$$

$$S_{\delta_3}^{R_0^P} = -\frac{\alpha_1\beta_2\left(\mu_h + \tau_2\right)\delta_3}{\left(\beta_1\left(\tau_2 + 1\right)\mu_h{}^2 + \left(\left(\tau_2 + 1\right)\left(\phi_1 + \delta_3\right)\beta_1 + \alpha_1\beta_2\right)\mu_h + \beta_2\alpha_1\tau_2\right)\left(\phi_1 + \delta_3 + \mu_h\right)} = -0.2966$$

$$S_{\mu_h}^{R_0^P} = -0.0001478.$$

2.7 Sensitivity Bar chart

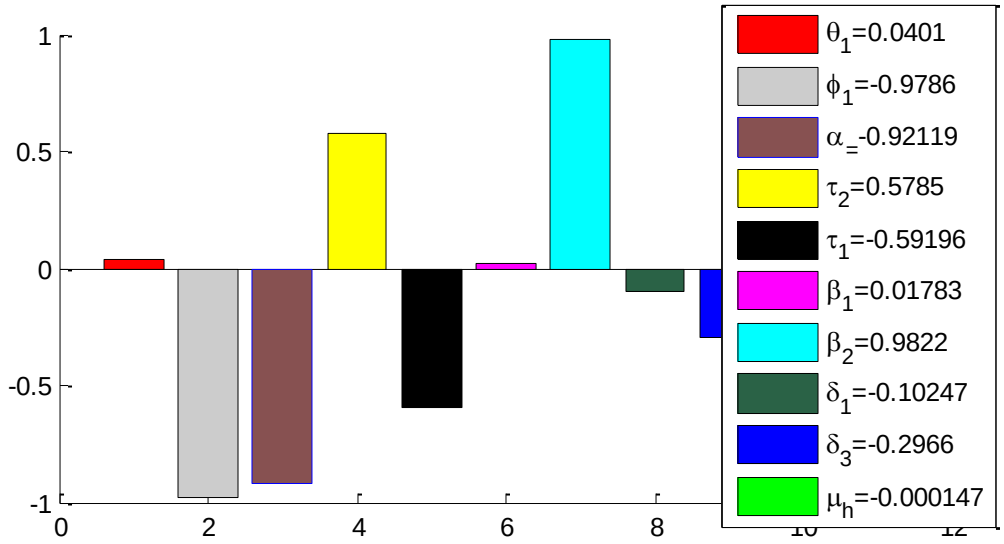


Fig.2: Sensitivity Bar chart

2.8 Interpretation of pneumonia Sensitivity Bar chart

As seen in the above figure 2, the figures show the sensitivity indices of basic reproduction number for pneumonia diseases. When the values of positive indices for pneumonia parameters increase, they become more able to lead to greater spread of the disease. When a parameter's value increases, the basic reproduction number gets larger and vice versa. Increases in these parameters help reduce disease in the community and bring down the number of infections likely to occur.

3.0 Pneumonia fractional order model

3.1 Existence and uniqueness results:

With the help of Banach's and Krassnoselskii's theorems, we can show that the model (1) does have at least one solution.

$$f_1 = \left(t, S_h, E_P, I_P, T_P, V_h, R_P, B\right) = \Lambda_h + \tau_2 V_h + \sigma R_p - \frac{\left(\beta_1 I_p + \beta_2 T_p + \beta_3 B\right)}{N_h} S_h - \left(\tau_1 + \mu_h\right) S_h,$$

$$f_2 = \left(t, S_h, E_P, I_P, T_P, V_h, R_P, B\right) = \frac{\left(\beta_1 I_p + \beta_2 T_p + \beta_3 B\right)}{N_h} S_h - \left(\theta_1 + \mu_h\right) E_P,$$

$$f_3 = \left(t, S_h, E_P, I_P, T_P, V_h, R_P, B\right) = \theta_1 E_P - \left(\alpha_1 + \delta_1 + \mu_h\right) I_P,$$

$$f_4 = \left(t, S_h, E_P, I_P, T_P, V_h, R_P, B\right) = \alpha_1 I_P - \left(\phi_1 + \delta_3 + \mu_h\right) T_P, \qquad (8)$$

$$f_5 = \left(t, S_h, E_P, I_P, T_P, V_h, R_P, B\right) = \tau_1 S_h - \left(\tau_2 + \mu_h\right) V_h,$$

$$f_6 = \left(t, S_h, E_P, I_P, T_P, V_h, R_P, B\right) = \phi_1 T_p - \left(\sigma + \mu_h\right) R_p,$$

$$f_7 = \left(t, S_h, E_P, I_P, T_P, V_h, R_P, B\right) = \Lambda_B - \mu_B B.$$

Where $S_h\left(0\right) = M_1, \mathrm{E}_P\left(0\right) = M_2, \mathrm{I}_P\left(0\right) = M_3, \mathrm{T}_P\left(0\right) = M_4, \mathrm{V}_h\left(0\right) = M_5, \mathrm{R}_P\left(0\right) = M_6$ and $B\left(0\right) = M_7$.

So our problem becomes:

$${}_0^{LC}D_t^{\alpha} A\left(t\right) = f_1\left(t, S_h, E_P, I_P, T_P, V_h, R_P, B\right),$$

$${}_0^{LC}D_t^{\alpha} B\left(t\right) = f_2\left(t, S_h, E_P, I_P, T_P, V_h, R_P, B\right),$$

$${}_0^{LC}D_t^{\alpha} C\left(t\right) = f_3\left(t, S_h, E_P, I_P, T_P, V_h, R_P, B\right),$$

$${}_0^{LC}D_t^{\alpha} \left(t\right) = f_4\left(t, S_h, E_P, I_P, T_P, V_h, R_P, B\right),$$

$${}_0^{LC}D_t^{\alpha} E\left(t\right) = f_5\left(t, S_h, E_P, I_P, T_P, V_h, R_P, B\right),$$

$${}_0^{LC}D_t^{\alpha} F\left(t\right) = f_6\left(t, S_h, E_P, I_P, T_P, V_h, R_P, B\right),$$

$${}_0^{LC}D_t^{\alpha} G\left(t\right) = f_7\left(t, S_h, E_P, I_P, T_P, V_h, R_P, B\right).$$

Where $S_h(0)=M_1, \mathrm{E}_P(0)=M_2, \mathrm{I}_P(0)=M_3, \mathrm{T}_P(0)=M_4, \mathrm{V}_h(0)=M_5, \mathrm{R}_P(0)=M_6$ an $B(0)=M_7$.

We consider

$$h(t)=\begin{pmatrix} S_h \\ E_P \\ I_P \\ T_P \\ V_h \\ R_P \\ B \end{pmatrix}, \qquad h_0(t)=\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \\ M_7 \end{pmatrix} \qquad \text{And} \quad F\left(t,h(t)\right)=\begin{pmatrix} f_1\left(t,h(t)\right) \\ f_2\left(t,h(t)\right) \\ f_3\left(t,h(t)\right) \\ f_4\left(t,h(t)\right) \\ f_5\left(t,h(t)\right) \\ f_6\left(t,h(t)\right) \\ f_7\left(t,h(t)\right) \end{pmatrix}.$$

Therefore, system (9) can be expressed as follows:

$$\begin{cases} {}_0^{LC}D_t^{\alpha}h(t)=F\left(t,h(t)\right), \ 0<\varepsilon\leq 1, \\ h(0)=h_0. \end{cases} \qquad \textbf{(10)}$$

The outcome of equation (10) is given by lemma 1 only if the right side is equal to zero at 0. That is:

$$h(t)+h_0 = XF\left(t,h(t)\right)+\bar{X}\int_0^t F\left(\varepsilon,h(\eta)\right)dn, \quad \text{...(ii)}$$

Where $X=\dfrac{1-\varepsilon}{K(\varepsilon)}$ and $\bar{X}=\dfrac{\varepsilon}{K(\varepsilon)}$.

Let us now define the Banach space $D=L[0,P]$ for more analysis by defining the norm of $D=L[0,P]$ on $0<t\leq P<\infty$.

$$\|h\|=\underset{t\in[0,P]}{Sup}\left\{\left|h(t):h\in D\right|\right\}.$$

Theorem (Krassnoselki fixed point theorem).

Let $D\subset Y$ be a convex and closed subset and there exist two operators Q_1 and Q_2 such that:

1. $Q_1h_1 + Q_2h_2 \in D$;
2. Q_2 is continuous, compact, and Q_1 is contraction;
3. There exists at least one fixed point h such that: $Q_1h + Q_2h = h$ holds.

(H_1) Let $K_F > 0$ be a constant, then

$$\left|F\left(t,h(t)\right) - F\left(t,\bar{h}(t)\right)\right| \leq K_F \left|h - \bar{h}\right|.$$

(H_2) for the two constant $A_F > 0$ and $B_F > 0$, one has $\left|K_F\left(t,h\right)\right| \leq \mathrm{A}_F \left|h\right| + B_F$,

Theorem 4.1

The problem of equation (2) has at least one solution if $GK_F < 1$.

Proof

Let define the set D as the set that is both compact and closed.

$D = \left\{h \in Y : \|h\| \leq r\right\}$. Let Q_1 and Q_2 be two operators, then:

$$Q_1 = h(t) = h_0 + GF\left(t,h(t)\right)$$

$$Q_2 = h(t) = h_0 + \bar{G}\int_0^t F\left(\varepsilon,h(\varepsilon)\right)d\varepsilon \qquad (12)$$

for the contraction condition of Q_1 defined in equation (12). Let $h, \bar{h} \in Y$, one has

$$\left\|Q_1h - Q_2\bar{h}\right\| = \underset{t\in(0,P)}{Sup}\left|F\left(t,h(t)\right) - F\left(t,\bar{h}(t)\right)\right| \qquad (13)$$

$\leq GK_F \left\|h - \bar{h}\right\|$, thus, Q_1 is contraction.

For compactness of Q_2, we consider

$$\left|Q_2h(t)\right| = \left|\bar{G}\int_0^t F\left(\eta,h(\eta)\right)d\eta\right|, \qquad (14)$$

$$\left|Q_2 h(t)\right| \le \left|\bar{G}\int_0^t F\left(\eta, h(\eta)\right)d\eta\right|.$$

We take the maximum of equation (14), which result to:

$$\left\|Q_2 h\right\| \le \bar{G}\,\underset{t\in[0,P]}{Sup}\int_0^t \left|F\left(\eta, h(\eta)\right)\right|d\eta, \qquad (15)$$

$$\left\|Q_2 h\right\| \le \bar{G}\,\underset{t\in[0,P]}{Sup}\left[A_F\left\|h\right\| + B_F\right]d\eta,$$

$$\left\|Q_2 h\right\| \le \bar{G}\,P\left(A_F r + B_F\right).$$

Then, Q_2 is bounded in equation (15).

Let the domain of t be $t_1 < t_2$, we have:

$$\left\|Q_2 h(t_2) - Q_2 h(t_1)\right\| = \left|\bar{G}\int_0^{t_2} F\left(\eta, h(\eta)\right)d\eta - \bar{G}\int_0^{t_1} F\left(\eta, h(\eta)\right)d\eta\right|$$

$$\left\|Q_2 h(t_2) - Q_2 h(t_1)\right\| = \left|\bar{G}\int_0^{t_2} F\left(\eta, h(\eta)\right)d\eta + \bar{G}\int_{t_1}^{0} F\left(\eta, h(\eta)\right)d\eta\right| \qquad (16)$$

$$\left\|Q_2 h(t_2) - Q_2 h(t_1)\right\| \le \bar{G}\left|\int_{t_1}^{0} F\left(\eta, h\right)d\eta\right|$$

$$\left\|Q_2 h(t_2) - Q_2 h(t_1)\right\| \le \bar{G}\left(A_F r + B_F\right).$$

Upon $t_2 \to t_1$, the right side of equation (16) tends to zero.

in addition, Q_2 is uniformly bounded, so

$$\left\|Q_2 h(t_2) - Q_2 h(t_2)\right\| \to 0.$$

So, all the assumptions of theorem 5 are valid, and the examined model (10) has at least one solution, since Q_2 is completely continuous.

Theorem 6

As a result of (H_1), if $\bar{G}F(1+P)<1$ holds, So, the only way to solve the problem given in equation (10) is by doing what is written. So, several different solutions can be found to the model (2).

Proof

Let $\sigma: Y \to Y$ be an operator defined by :

$$\sigma h(t) = h_0 + GF\left(t, h(t)\right) + \bar{G}\int_0^{t_1} F(\eta, h)d\eta.$$

Let $h, \bar{h} \in Y$, then,

$$\left\|\sigma(h) - \sigma\left(\bar{h}\right)\right\| = \underset{t\in[0,P]}{Sup}\left(\sigma(h)(t) - \sigma\left(\bar{h}\right)(t)\right),$$

$$\leq \underset{t\in[0,P]}{Sup}\left|F\left(t, h(t)\right) - F\left(t, \bar{h}(t)\right) - F\left(\eta, \bar{h}(\eta)\right)\right| d\eta,$$

$$\leq \bar{G}K_F\left\|h - \bar{h}\right\| + GK_F P\left\|h - \bar{h}\right\|,$$

$$\left\|\sigma(h) - \sigma\left(\bar{h}\right)\right\| \leq \bar{G}K_F(1+P)\left\|h - \left(\bar{h}\right)\right\| \qquad (17)$$

As a result, the most possible outcome for problem (10) is only one option, meaning that model 2 has only one solution.

3.2 Developing a generic algorithm to solve the model under consideration

Let $K(\sigma)=1$, the use of Laplace transform gives us a series form of the solution handy. Therefore, the algorithm in Figure 1 can be created.

$$\frac{SL\left[S_h(t)\right] - S(0)}{S + \sigma(1-S)} = \Lambda_h + \tau_2 V_h + \sigma R_p - \frac{\left(\beta_1 I_p + \beta_2 T_p + \beta_3 B\right)}{N_h} S_h - (\tau_1 + \mu_h) S_h,$$

$$\frac{SL\left[E_P(t)\right] - S(0)}{S + \sigma(1-S)} = \frac{\left(\beta_1 I_p + \beta_2 T_p + \beta_3 B\right)}{N_h} S_h - (\theta_1 + \mu_h) E_P,$$

$$\frac{S\,L\left[I_P(t)\right]-S(0)}{S+\sigma(1-S)}=\theta_1 E_P-(\alpha_1+\delta_1+\mu_h)I_P, \qquad (18)$$

$$\frac{S\,L\left[T_P(t)\right]-S(0)}{S+\sigma(1-S)}=\alpha_1 I_P-(\phi_1+\delta_3+\mu_h)T_P,$$

$$\frac{S\,L\left[V_h(t)\right]-S(0)}{S+\sigma(1-S)}=\tau_1 S_h-(\tau_2+\mu_h)V_h,$$

$$\frac{S\,L\left[R_p(t)\right]-S(0)}{S+\sigma(1-S)}=\phi_1 T_p-(\sigma+\mu_h)R_p,$$

$$\frac{S\,L\left[B(t)\right]-S(0)}{S+\sigma(1-S)}=\Lambda_B-\mu_B B.$$

From (18) we obtained:

$$L\left[S_h(t)\right]=\frac{S_h(0)}{S}+\frac{S+\sigma(1-S)}{S}L\left[\Lambda_h+\tau_2 V_h+\sigma R_p-\frac{\left(\beta_1 I_p+\beta_2 T_p+\beta_3 B\right)}{N_h}S_h-(\tau_1+\mu_h)S_h\right],$$

$$L\left[E_P(t)\right]=\frac{E_P(0)}{S}+\frac{S+\sigma(1-S)}{S}L\left[\frac{\left(\beta_1 I_p+\beta_2 T_p+\beta_3 B\right)}{N_h}S_h-(\theta_1+\mu_h)E_P\right],$$

$$L\left[I_P(t)\right]=\frac{I_P(0)}{S}+\frac{S+\sigma(1-S)}{S}L\left[\theta_1 E_P-(\alpha_1+\delta_1+\mu_h)I_P\right],$$

$$L\left[T_P(t)\right]=\frac{T_P(0)}{S}+\frac{S+\sigma(1-S)}{S}L\left[\alpha_1 I_P-(\phi_1+\delta_3+\mu_h)T_P\right], \qquad (19)$$

$$L\left[V_h(t)\right]=\frac{V_h(0)}{S}+\frac{S+\sigma(1-S)}{S}L\left[\tau_1 S_h-(\tau_2+\mu_h)V_h\right],$$

$$L\left[R_p(t)\right]=\frac{R_p(0)}{S}+\frac{S+\sigma(1-S)}{S}L\left[\phi_1 T_p-(\sigma+\mu_h)R_p\right],$$

$$L\left[B(t)\right]=\frac{B(0)}{S}+\frac{S+\sigma(1-S)}{S}L\left[\Lambda_B-\mu_B B\right].$$

Applying the initial conditions

$S_h(0)=M_1, \mathrm{E}_P(0)=M_2, \mathrm{I}_P(0)=M_3, \mathrm{T}_P(0)=M_4, \mathrm{V}_h(0)=M_5, \mathrm{R}_P(0)=M_6$ and $B(0)=M_7$

.

we have:

$$L\left[S_h(t)\right]=\frac{M_1}{S}+\frac{S+\sigma(1-S)}{S}L\left[\Lambda_h+\tau_2 V_h+\sigma R_p-\frac{\left(\beta_1 I_p+\beta_2 T_p+\beta_3 B\right)}{N_h}S_h-(\tau_1+\mu_h)S_h\right],$$

$$L\left[E_P(t)\right]=\frac{M_2}{S}+\frac{S+\sigma(1-S)}{S}L\left[\frac{\left(\beta_1 I_p+\beta_2 T_p+\beta_3 B\right)}{N_h}S_h-(\theta_1+\mu_h)E_P\right],$$

$$L\left[I_P(t)\right]=\frac{M_3}{S}+\frac{S+\sigma(1-S)}{S}L\left[\theta_1 E_P-(\alpha_1+\delta_1+\mu_h)I_P\right],$$

$$L\left[T_P(t)\right]=\frac{M_4}{S}+\frac{S+\sigma(1-S)}{S}L\left[\alpha_1 I_P-(\phi_1+\delta_3+\mu_h)T_P\right], \qquad (20)$$

$$L\left[V_h(t)\right]=\frac{M_5}{S}+\frac{S+\sigma(1-S)}{S}L\left[\tau_1 S_h-(\tau_2+\mu_h)V_h\right],$$

$$L\left[R_p(t)\right]=\frac{M_6}{S}+\frac{S+\sigma(1-S)}{S}L\left[\phi_1 T_p-(\sigma+\mu_h)R_p\right],$$

$$L\left[B(t)\right]=\frac{M_7}{S}+\frac{S+\sigma(1-S)}{S}L\left[\Lambda_B-\mu_B B\right].$$

We assume that the solution takes the form of an infinite series, as presented below:

$$S_h(t)=\sum_{n=0}^{\infty}(S_{hn})(t), \mathrm{E}_P(t)=\sum_{n=0}^{\infty}(E_{Pn})(t), \mathrm{I}_P(t)=\sum_{n=0}^{\infty}(I_{Pn})(t), \mathrm{T}_P(t)=\sum_{n=0}^{\infty}(T_{Pn})(t),$$
$$\mathrm{V}_h(t)=\sum_{n=0}^{\infty}(\mathrm{V}_{hn})(t), \mathrm{R}_P(t)=\sum_{n=0}^{\infty}(R_{Pn})(t), B(t)=\sum_{n=0}^{\infty}(B_n)(t).$$

The non linear terms $\mathrm{I}_P S_h$, $\mathrm{T}_P S_h$ and BS_h are decomposes in terms of Adomian polynomial as:

$\mathrm{I}_P(t)S_h(t)=\sum_{n=0}^{\infty}T_n(t)$, where

$$T_n = \frac{1}{\tau(n+1)} \frac{d^n}{d\lambda^n} \left[\left(\sum_{K=0}^{n} \lambda^K \mathrm{I}_{PK} \right) \left(\sum_{K=0}^{n} \lambda^K S_{hK} \right) \right] \Bigg|_{\lambda=0} .$$

if $n=0: T_0 = S_{h0}(t) I_{P0}(t).$

$n=1: T_1 = S_{h0}(t) I_{P1}(t) + S_{h1}(t) I_{P0}(t).$

$n=2: T_2 = S_{h0}(t) I_{P0}(t) + S_{h1}(t) I_{P1}(t) + S_{h2}(t) I_{P0}(t).$

$n=3: T_3 = S_{h0}(t) I_{P3}(t) + S_{h1}(t) I_{P2}(t) + S_{h2}(t) I_{P2}(t) + S_{h3}(t) I_{P1}(t) + S_{h4}(t) I_{P0}(t).$

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$n=n: T_n = S_{hn}(t) I_{Pn}(t) + S_{h1}(t) I_{P(n-1)}(t) + \ldots + S_{h(n-1)}(t) I_{P1}(t) + S_{hn}(t) I_{P0}(t).$

Similarly

$T_P(t) S_h(t) = \sum_{n=0}^{\infty} U_n(t),$ where

$$U_n = \frac{1}{\tau(n+1)} \frac{d^n}{d\lambda^n} \left[\left(\sum_{K=0}^{n} \lambda^K \mathrm{T}_{PK} \right) \left(\sum_{K=0}^{n} \lambda^K S_{hK} \right) \right] \Bigg|_{\lambda=0} .$$

if $n=0: U_0 = S_{h0}(t) T_{P0}(t).$

$n=1: \mathrm{U}_1 = S_{h0}(t) T_{P1}(t) + S_{h1}(t) T_{P0}(t).$

$n=2: U_2 = S_{h0}(t) T_{P0}(t) + S_{h1}(t) T_{P1}(t) + S_{h2}(t) T_{P0}(t).$

$n=3: \mathrm{U}_3 = S_{h0}(t) T_{P3}(t) + S_{h1}(t) T_{P2}(t) + S_{h2}(t) T_{P2}(t) + S_{h3}(t) T_{P1}(t) + S_{h4}(t) T_{P0}(t).$

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$$n=n:\mathrm{U}_n=S_{hn}(t)T_{Pn}(t)+S_{h1}(t)T_{P(n-1)}(t)+\ldots+S_{h(n-1)}(t)T_{P1}(t)+S_{hn}(t)T_{P0}(t).$$

Also

$$B(t)S_h(t)=\sum_{n=0}^{\infty}W_n(t),\ \text{where}$$

$$W_n=\frac{1}{\tau(n+1)}\frac{d^n}{d\lambda^n}\left[\left(\sum_{K=0}^{n}\lambda^K B_K\right)\left(\sum_{K=0}^{n}\lambda^K S_{hK}\right)\right]\Bigg|_{\lambda=0}.$$

if $n=0:\mathrm{W}_0=S_{h0}(t)W_{P0}(t).$

$$n=1:W_1=S_{h0}(t)B_1(t)+S_{h1}(t)B_0(t).$$

$$n=2:W_2=S_{h0}(t)B_0(t)+S_{h1}(t)B_1(t)+S_{h2}(t)B_0(t).$$

$$n=3:\mathrm{W}_3=S_{h0}(t)B_3(t)+S_{h1}(t)B_2(t)+S_{h2}(t)B_2(t)+S_{h3}(t)B_1(t)+S_{h4}(t)B_0(t).$$

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$$n=n:W_n=S_{hn}(t)B_n(t)+S_{h1}(t)B_{(n-1)}(t)+\ldots+B_{(n-1)}(t)B_1(t)+S_{hn}(t)B_0(t).$$

taking into consideration these values, model develops

$$L\left[\sum_{K=0}^{\infty}S_h(t)\right]=\frac{M_1}{S}+\frac{S+\sigma(1-S)}{S}L\left[\begin{array}{l}\Lambda_h+\tau_2\sum_{K=0}^{\infty}V_h+\sigma\sum_{K=0}^{\infty}R_p-\frac{\left(\beta_1\sum_{K=0}^{\infty}I_p+\beta_2\sum_{K=0}^{\infty}T_p+\beta_3\sum_{K=0}^{\infty}B\right)}{N_h}S_h\\-(\tau_1+\mu_h)\sum_{K=0}^{\infty}S_h\end{array}\right],$$

$$L\left[\sum_{K=0}^{\infty}E_P(t)\right]=\frac{M_2}{S}+\frac{S+\sigma(1-S)}{S}L\left[\begin{array}{l}\frac{\left(\beta_1\sum_{K=0}^{\infty}I_p+\beta_2\sum_{K=0}^{\infty}T_p+\beta_3\sum_{K=0}^{\infty}B\right)}{N_h}S_h-(\theta_1+\mu_h)\sum_{K=0}^{\infty}E_P\end{array}\right],$$

$$L\left[\sum_{K=0}^{\infty} I_P(t)\right]=\frac{M_3}{S}+\frac{S+\sigma(1-S)}{S}L\left[\theta_1\sum_{K=0}^{\infty}E_P(t)-(\alpha_1+\delta_1+\mu_h)\sum_{K=0}^{\infty}I_P(t)\right],$$

$$L\left[\sum_{K=0}^{\infty} T_P(t)\right]=\frac{M_4}{S}+\frac{S+\sigma(1-S)}{S}L\left[\alpha_1\sum_{K=0}^{\infty}I_P(t)-(\phi_1+\delta_3+\mu_h)\sum_{K=0}^{\infty}T_P(t)\right], \quad (21)$$

$$L\left[\sum_{K=0}^{\infty} V_h(t)\right]=\frac{M_5}{S}+\frac{S+\sigma(1-S)}{S}L\left[\tau_1\sum_{K=0}^{\infty}S_h(t)-(\tau_2+\mu_h)\sum_{K=0}^{\infty}V_h(t)\right],$$

$$L\left[\left[\sum_{K=0}^{\infty} R_p(t)\right]\right]=\frac{M_6}{S}+\frac{S+\sigma(1-S)}{S}L\left[\phi_1\sum_{K=0}^{\infty}T_p(t)-(\sigma+\mu_h)\sum_{K=0}^{\infty}R_p(t)\right],$$

$$L\left[\sum_{K=0}^{\infty} B(t)\right]=\frac{M_7}{S}+\frac{S+\sigma(1-S)}{S}L\left[\Lambda_B-\mu_B\sum_{K=0}^{\infty}B(t)\right].$$

Comparing the terms of equation (21), the following complications arises.

Cases 1: If we let n=0, then

$$L\left[S_{h0}(t)\right]=\frac{M_1}{S}+\frac{S+\sigma(1-S)}{S}L(\Lambda_h),$$

$$L\left[E_{P0}(t)\right]=\frac{M_2}{S},$$

$$L\left[I_{P0}(t)\right]=\frac{M_3}{S},$$

$$L\left[T_{P0}(t)\right]=\frac{M_4}{S}, \quad (22)$$

$$L\left[V_{h0}(t)\right]=\frac{M_5}{S},$$

$$L\left[R_{P0}(t)\right]=\frac{M_6}{S},$$

$$L\left[B_0(t)\right]=\frac{M_7}{S}.$$

We take the inverse Laplace transform, we obtained:

$$S_{h0}(t) = M_1 + S + (\Lambda_h)\left[1+\sigma(t-1)\right],$$

$$E_{P0}(t) = M_2,$$

$$I_{P0}(t) = M_3,$$

$$T_{P0}(t) = M_4, \qquad (23)$$

$$V_{h0}(t) = M_5,$$

$$R_{P0}(t) = M_6,$$

$$B_0(t) = M_7.$$

Cases 2: If we let n=1, then:

$$L\left[S_{h1}(t)\right] = \frac{S+\sigma(1-S)}{S} L\left[-\frac{(\beta_1 T_0 + \beta_2 U_0 + \beta_3 W_0)}{N_h} S_{h0} - (\tau_1 + \mu_h) S_{h0}\right],$$

$$L\left[E_{P1}(t)\right] = \frac{S+\sigma(1-S)}{S} L\left[\frac{(\beta_1 T_0 + \beta_2 U_0 + \beta_3 W_0)}{N_h} S_{h0} - (\theta_1 + \mu_h) E_{P0}\right],$$

$$L\left[I_{P1}(t)\right] = \frac{S+\sigma(1-S)}{S} L\left[\theta_1 E_{P0} - (\alpha_1 + \delta_1 + \mu_h) I_{P0}\right],$$

$$L\left[T_{P1}(t)\right] = \frac{S+\sigma(1-S)}{S} L\left[\alpha_1 I_{P0} - (\phi_1 + \delta_3 + \mu_h) T_{P0}\right], \qquad (24)$$

$$L\left[V_{h1}(t)\right] = \frac{S+\sigma(1-S)}{S} L\left[\tau_1 S_{h0} - (\tau_2 + \mu_h) V_{h0}\right],$$

$$L\left[R_{p1}(t)\right] = \frac{S+\sigma(1-S)}{S} L\left[\phi_1 T_{p0} - (\sigma + \mu_h) R_{p0}\right],$$

$$L\left[B_1(t)\right] = \frac{S+\sigma(1-S)}{S} L\left[-\mu_B B_0\right].$$

We take the inverse Laplace transform we have:

$$\left[S_{h1}(t)\right]=\left[-\frac{\left(\beta_1 I_{P0}+\beta_2 T_{P0}+\beta_3 B_0\right)}{N_h}S_{h0}-\left(\tau_1+\mu_h\right)S_{h0}\right]\left[1+\sigma\left(t-1\right)\right],$$

$$\left[E_{P1}(t)\right]=\left[\frac{\left(\beta_1 I_{P0}+\beta_2 T_{P0}+\beta_3 B_0\right)}{N_h}S_{h0}-\left(\theta_1+\mu_h\right)E_{P0}\right]\left[1+\sigma\left(t-1\right)\right],$$

$$\left[I_{P1}(t)\right]=\left[\theta_1 E_{P0}-\left(\alpha_1+\delta_1+\mu_h\right)I_{P0}\right]\left[1+\sigma\left(t-1\right)\right],$$

$$\left[T_{P1}(t)\right]=\left[\alpha_1 I_{P0}-\left(\phi_1+\delta_3+\mu_h\right)T_{P0}\right]\left[1+\sigma\left(t-1\right)\right], \qquad (25)$$

$$\left[V_{h1}(t)\right]=\left[\tau_1 S_{h0}-\left(\tau_2+\mu_h\right)V_{h0}\right]\left[1+\sigma\left(t-1\right)\right],$$

$$\left[R_{p1}(t)\right]=\frac{S+\sigma\left(1-S\right)}{S}\left[\phi_1 T_{p0}-\left(\sigma+\mu_h\right)R_{p0}\right]\left[1+\sigma\left(t-1\right)\right],$$

$$\left[B_1(t)\right]=\left[-\mu_B B_0\right]\left[1+\sigma\left(t-1\right)\right].$$

Case 3: if n=2 we have:

$$L\left[S_{h2}(t)\right]=\frac{S+\sigma\left(1-S\right)}{S}L\left[-\frac{\left(\beta_1 T_1+\beta_2 U_1+\beta_3 W_1\right)}{N_h}S_{h1}-\left(\tau_1+\mu_h\right)S_{h1}\right],$$

$$L\left[E_{P2}(t)\right]=\frac{S+\sigma\left(1-S\right)}{S}L\left[\frac{\left(\beta_1 T_1+\beta_2 U_1+\beta_3 W_1\right)}{N_h}S_{h1}-\left(\theta_1+\mu_h\right)E_{P1}\right],$$

$$L\left[I_{P2}(t)\right]=\frac{S+\sigma\left(1-S\right)}{S}L\left[\theta_1 E_{P1}-\left(\alpha_1+\delta_1+\mu_h\right)I_{P1}\right],$$

$$L\left[T_{P2}(t)\right]=\frac{S+\sigma\left(1-S\right)}{S}L\left[\alpha_1 I_{P1}-\left(\phi_1+\delta_3+\mu_h\right)T_{P1}\right], \qquad (26)$$

$$L\left[V_{h2}(t)\right]=\frac{S+\sigma\left(1-S\right)}{S}L\left[\tau_1 S_{h1}-\left(\tau_2+\mu_h\right)V_{h1}\right],$$

$$L\left[R_{p2}(t)\right]=\frac{S+\sigma\left(1-S\right)}{S}L\left[\phi_1 T_{p1}-\left(\sigma+\mu_h\right)R_{p1}\right],$$

$$L\left[B_2(t)\right]=\frac{S+\sigma\left(1-S\right)}{S}L\left[-\mu_B B_1\right].$$

Taking the inverse Laplace transform we have:

$$\left[S_{h2}(t)\right]=\left[-\frac{\left(\beta_1 I_{P1}+\beta_2 T_{P1}+\beta_3 B_1\right)}{N_h}S_{h1}-\left(\tau_1+\mu_h\right)S_{h1}\right]\left[1+\sigma\left(t-1\right)\right],$$

$$\left[E_{P2}(t)\right]=\left[\frac{\left(\beta_1 I_{P1}+\beta_2 T_{P1}+\beta_3 B_1\right)}{N_h}S_{h1}-\left(\theta_1+\mu_h\right)E_{P1}\right]\left[1+\sigma\left(t-1\right)\right],$$

$$\left[I_{P2}(t)\right]=\left[\theta_1 E_{P1}-\left(\alpha_1+\delta_1+\mu_h\right)I_{P1}\right]\left[1+\sigma\left(t-1\right)\right],$$

$$\left[T_{P2}(t)\right]=\left[\alpha_1 I_{P1}-\left(\phi_1+\delta_3+\mu_h\right)T_{P1}\right]\left[1+\sigma\left(t-1\right)\right], \qquad (27)$$

$$\left[V_{h2}(t)\right]=\left[\tau_1 S_{h1}-\left(\tau_2+\mu_h\right)V_{h1}\right]\left[1+\sigma\left(t-1\right)\right],$$

$$\left[R_{p2}(t)\right]=\frac{S+\sigma\left(1-S\right)}{S}\left[\phi_1 T_{p1}-\left(\sigma+\mu_h\right)R_{p1}\right]\left[1+\sigma\left(t-1\right)\right],$$

$$\left[B_2(t)\right]=\left[-\mu_B B_1\right]\left[1+\sigma\left(t-1\right)\right].$$

Substituting (25) into (27) we have:

$$\begin{aligned}S_{h2}(t)=&-\frac{\left(\beta_1\left[\left[\theta_1 E_{P0}-\left(\alpha_1+\delta_1+\mu_h\right)I_{P0}\right]\left[1+\sigma\left(t-1\right)\right]\right]S_{h1}\right)}{N_h}\\&+\left[\begin{array}{l}\dfrac{\beta_2\left[\alpha_1 I_{P0}-\left(\phi_1+\delta_3+\mu_h\right)T_{P0}\right]\left[1+\sigma\left(t-1\right)\right]S_{h1}}{N_h}\\+\dfrac{\beta_3\left[\left[-\mu_B B_0\right]\left[1+\sigma\left(t-1\right)\right]\right]S_{h1}}{N_h}\end{array}\right]\\&-\left(\tau_1+\mu_h\right)S_{h1}\left[1+\sigma\left(t-1\right)\right],\end{aligned}$$

$$\left[E_{P2}(t)\right]=\frac{\left(\beta_1\left[\theta_1 E_{P0}-\left(\alpha_1+\delta_1+\mu_h\right)I_{P0}\right]\left[1+\sigma(t-1)\right]\right)}{N_h}$$
$$+\frac{\beta_2\left(\left[\alpha_1 I_{P0}-\left(\phi_1+\delta_3+\mu_h\right)T_{P0}\right]\left[1+\sigma(t-1)\right]\right)}{N_h}$$
$$+\frac{\beta_3\left[-\mu_B B_0\right]\left[1+\sigma(t-1)\right]S_{h1}-\left(\theta_1+\mu_h\right)}{N_h}$$
$$\left[\frac{\left(\beta_1 I_{P0}+\beta_2 T_{P0}+\beta_3 B_0\right)}{N_h}S_{h0}-\left(\theta_1+\mu_h\right)E_{P0}\right]$$
$$\left[1+\sigma(t-1)\right]\left[1+\sigma(t-1)\right],$$

$$I_{P2}(t)=\theta_1\left[\frac{\left(\beta_1 I_{P0}+\beta_2 T_{P0}+\beta_3 B_0\right)}{N_h}S_{h0}-\left(\theta_1+\mu_h\right)E_{P0}\right]\left[1+\sigma(t-1)\right] \tag{27}$$
$$-\left(\alpha_1+\delta_1+\mu_h\right)\left[\theta_1 E_{P0}-\left(\alpha_1+\delta_1+\mu_h\right)I_{P0}\right]\left[1+\sigma(t-1)\right]\left[1+\sigma(t-1)\right],$$

$$T_{P2}(t)=\begin{bmatrix}\alpha_1\left[\theta_1 E_{P0}-\left(\alpha_1+\delta_1+\mu_h\right)I_{P0}\right]\left[1+\sigma(t-1)\right]\\-\left(\phi_1+\delta_3+\mu_h\right)\left[\alpha_1 I_{P0}-\left(\phi_1+\delta_3+\mu_h\right)T_{P0}\right]\left[1+\sigma(t-1)\right]\end{bmatrix}\left[1+\sigma(t-1)\right],$$

$$V_{h2}(t)=\begin{bmatrix}\tau_1\left[-\frac{\left(\beta_1 I_{P0}+\beta_2 T_{P0}+\beta_3 B_0\right)}{N_h}S_{h0}-\left(\tau_1+\mu_h\right)S_{h0}\right]\left[1+\sigma(t-1)\right]\\-\left(\tau_2+\mu_h\right)\left[\tau_1 S_{h0}-\left(\tau_2+\mu_h\right)V_{h0}\right]\left[1+\sigma(t-1)\right]\end{bmatrix}\left[1+\sigma(t-1)\right],$$

$$R_{p2}(t)=\frac{S+\sigma(1-S)}{S}\begin{bmatrix}\phi_1\left[\alpha_1 I_{P0}-\left(\phi_1+\delta_3+\mu_h\right)T_{P0}\right]\left[1+\sigma(t-1)\right]\\-\left(\sigma+\mu_h\right)\frac{S+\sigma(1-S)}{S}\left[\phi_1 T_{p0}-\left(\sigma+\mu_h\right)R_{p0}\right]\left[1+\sigma(t-1)\right]\end{bmatrix}\left[1+\sigma(t-1)\right],$$

$$B_2(t)=\left[-\mu_B\left[-\mu_B B_0\right]\left[1+\sigma(t-1)\right]\right]\left[1+\sigma(t-1)\right].$$

The technique can be used to discover further terms in the solution. For that reason, we find the solution as follow:

$$S_h(t)=S_{h0}(t)+S_{h1}(t)+S_{h2}(t)+\ldots$$

$$E_P(t)=E_{P0}(t)+E_{P1}(t)+E_{P2}(t)+\ldots$$

$$I_P(t) = I_{P0}(t) + I_{P1}(t) + I_{P2}(t) + \ldots$$

$$T_P(t) = T_{P0}(t) + T_{P1}(t) + T_{P2}(t) + \ldots$$

$$V_h(t) = V_{h0}(t) + V_{h1}(t) + V_{h2}(t) + \ldots$$

$$R_P(t) = R_{P0}(t) + R_{P1}(t) + R_{P2}(t) + \ldots$$

$$B(t) = B_0(t) + B_1(t) + B_2(t) + \ldots.$$

4.0 Numerical Simulation

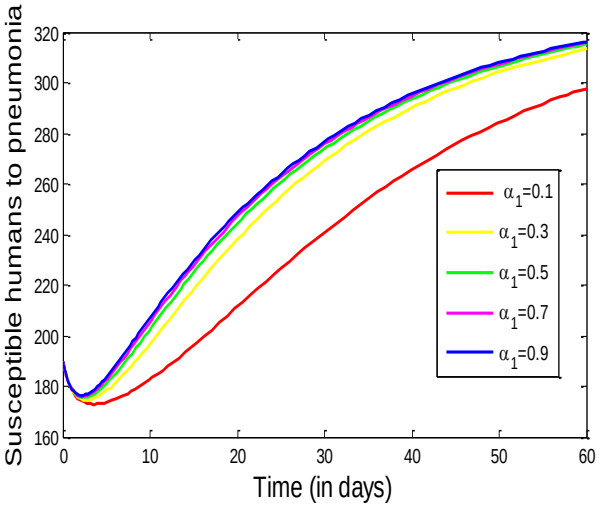


Fig.3a: Susceptible humans to pneumonia

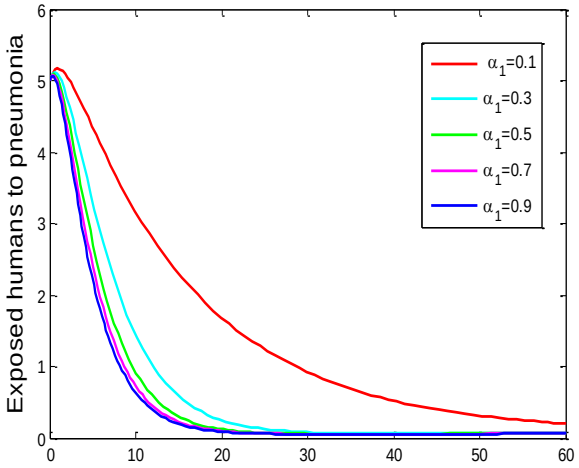


Fig.3b: Exposed humans to pneumonia

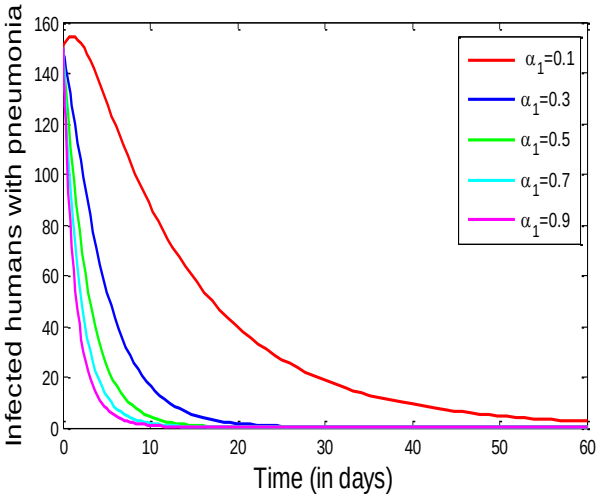


Fig.3c: Infected humans with pneumonia

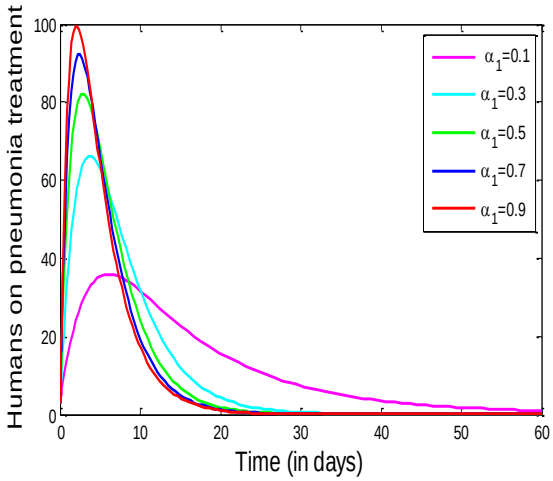


Fig.3d: Humans on pneumonia treatment

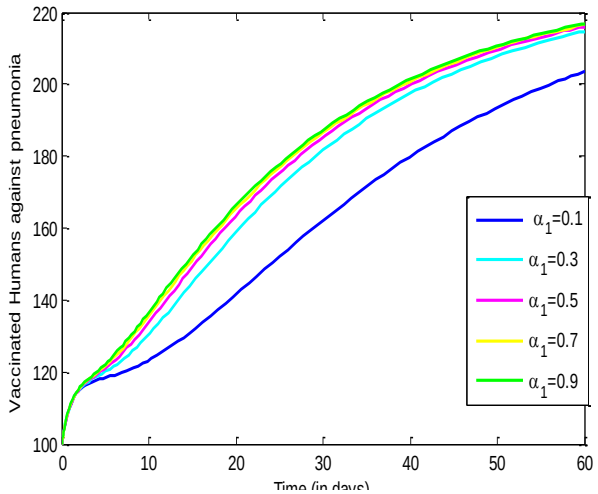


Fig.3e: Vaccinated humans against pneumonia

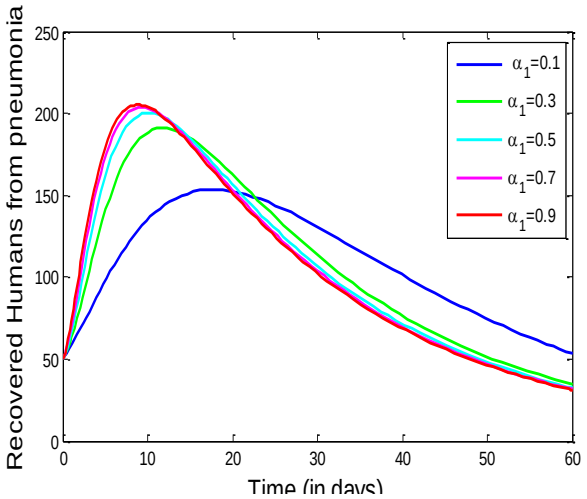


Fig.3 f: Recovered Humans from pneumonia

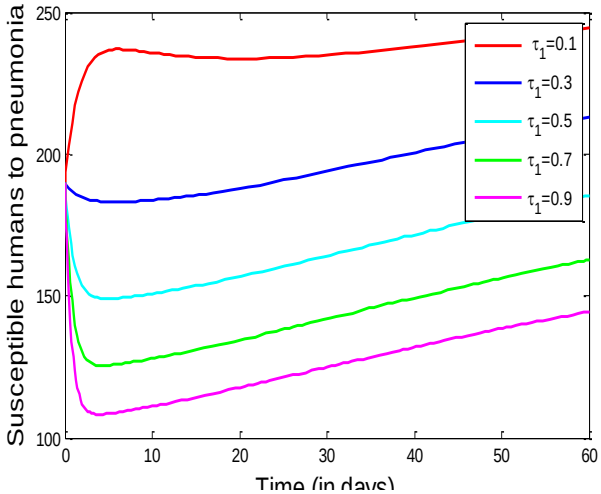


Fig.3g: Susceptible humans to pneumonia

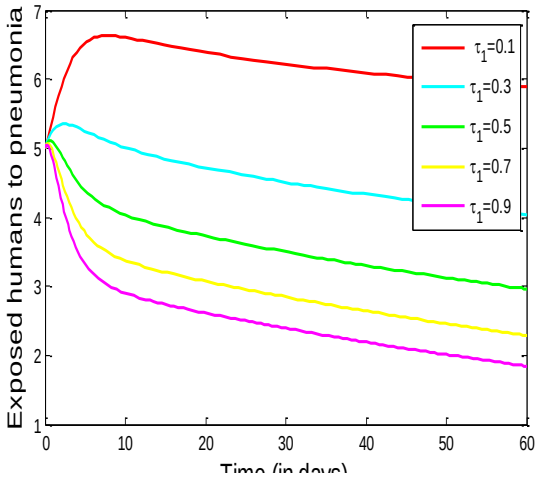
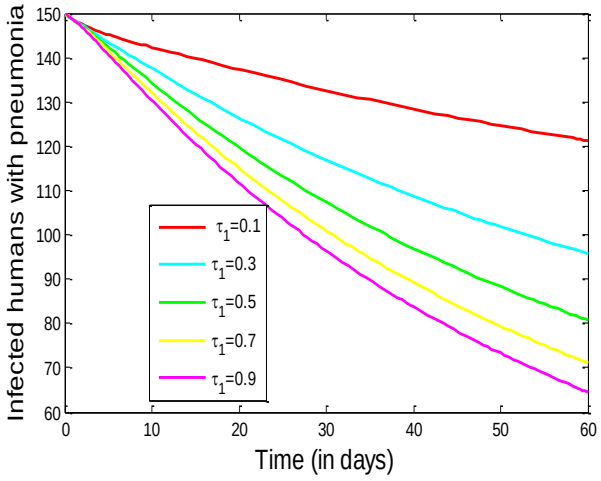


Fig.3h: Exposed humans to pneumonia



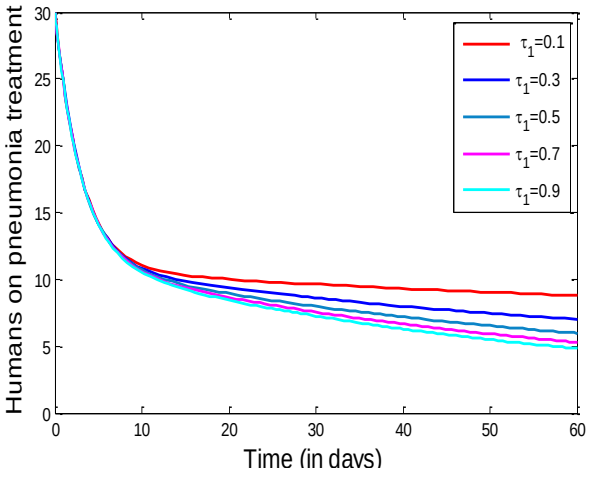


Fig.3j: Humans on pneumonia treatment

Fig.3i: Infected humans with pneumonia

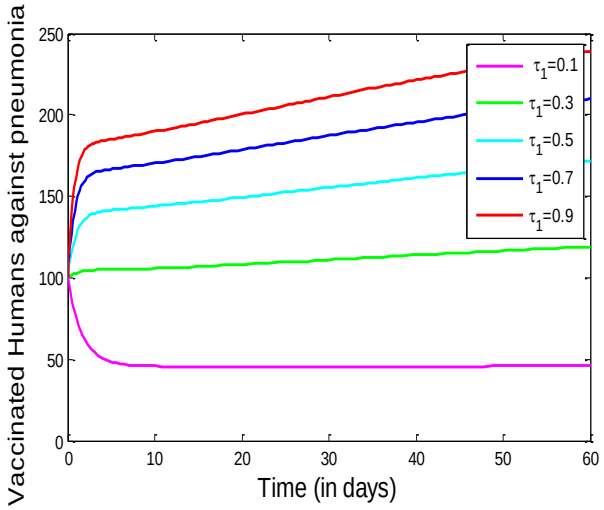


Fig.3k: Vaccinated humans against pneumonia

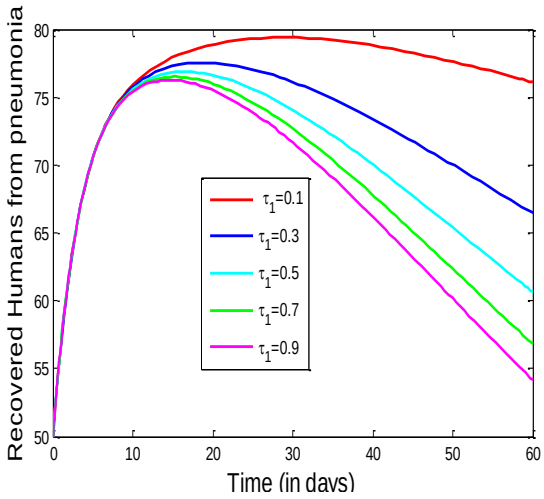


Fig.3l: Recovered Humans from pneumonia

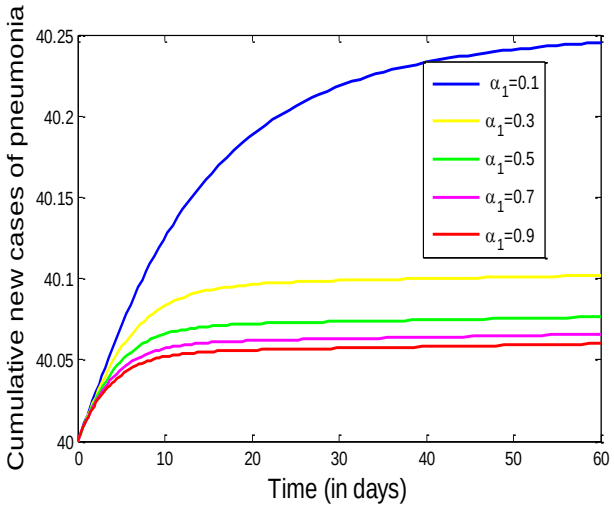


Fig.3m: Cumulative new cases of pneumonia

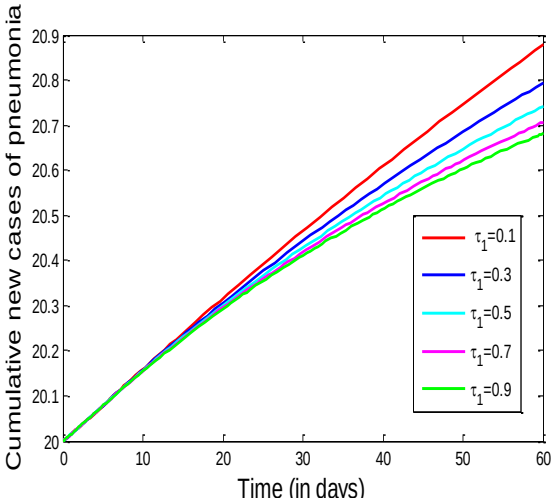
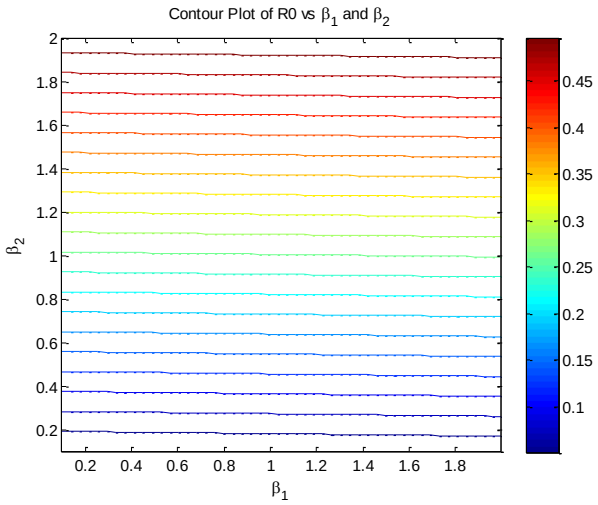


Fig.3n: Cumulative new cases of pneumonia



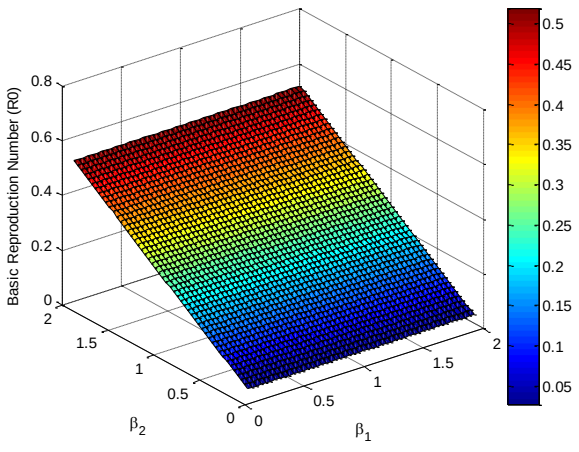


Fig.4a: Contour plot showing the impact of β_1 *and* β_2 **on** R_0^P

Fig.4b: Surface plot showing the impact of β_1 *and* β_2 **on** R_0^P

(3a) displays the impact of treatment rate (α_1) on susceptible humans to pneumonia. It appears that giving treatment to more patients at the rate (α_1) leads to rising numbers of humans who can develop pneumonia. (3b) displays the impact of treatment rate (α_1) on Exposed humans to pneumonia. It appears that giving treatment to more patients at the rate (α_1) leads to decrease in the numbers of humans exposed to pneumonia decreases. (3c) displays the impact of treatment rate (α_1) on infected humans with pneumonia. It appears that giving treatment to more patients at the rate (α_1) leads to decrease in the numbers of humans infected to pneumonia decreases.

(3d) displays the impact of treatment rate (α_1) on humans' pneumonia treatment. It appears that giving treatment to more patients at the rate (α_1) leads to increase in the numbers of humans on treatment of pneumonia. (3e) displays the impact of treatment rate (α_1) on vaccinated humans against pneumonia. It appears that giving treatment to more patients at the rate (α_1) leads to increase in the numbers of vaccinated humans against pneumonia. (3f) displays the impact of treatment rate (α_1) on Recovered humans from pneumonia. It appears that giving treatment to more patients at the rate (α_1) leads to increase in the numbers of Recovered humans from pneumonia. (3g) displays the impact of vaccination rate (τ_1) on susceptible humans to pneumonia. It appears that vaccinating more patients at the rate (τ_1) leads to rising numbers of humans who can develop pneumonia. (3h) displays the impact of vaccination rate (τ_1) on Exposed humans to pneumonia. It appears that vaccinating more patients at the rate (τ_1) leads to decrease in the numbers of humans exposed to pneumonia decreases. (3i) displays the impact of vaccination rate (τ_1) on infected humans with pneumonia. It appears that vaccinating more patients at the rate (τ_1) leads to decrease in the numbers of humans infected to pneumonia decreases. (3j) displays the impact of vaccination rate (τ_1) on humans' pneumonia treatment. It appears that vaccinating more patients at the rate (τ_1) leads to increase in the numbers of humans on treatment of pneumonia. (3k) displays the impact of vaccination rate (τ_1) on vaccinated humans against pneumonia. It appears that vaccinating more patients at the rate (τ_1) leads to increase in the numbers of vaccinated humans against pneumonia.

(3l) displays the impact of vaccination rate (τ_1) on Recovered humans from pneumonia. It appears that vaccinating more patients at the rate (τ_1) leads to increase in the numbers of Recovered humans from pneumonia. (3m) displays the impact of treatment rate (α_1) on Recovered humans from pneumonia. It appears that increasing treatment rate (α_1) leads to decrease in the cumulative new cases of pneumonia. (3n) displays the impact of vaccination rate (τ_1) on Recovered humans from pneumonia. It appears that increasing vaccination rate (τ_1) leads to decrease in the cumulative new cases of pneumonia. (4a) displays contour plot of $\beta_1 \, and \, \beta_2$ concerning R_0. The graph's figures demonstrate that the maximum value is 0.6 and this means that by varying the parameters $\beta_1 \, and \, \beta_2$, the transmission rate will be less than one (1) when measured. When these factors rise, it demonstrates that there is likely to be a big outbreak of pneumonia. (4b) It was shown that the basic reproduction number R_0 decreases and goes below one (1) when $\beta_1 \, and \, \beta_2$ goes down. If $\beta_1 \, and \, \beta_2$ decrease, it would help to lower the effect of pneumonia among people. Lack of suitable measures on $\beta_1 \, and \, \beta_2$ will make pneumonia more present in the community.

5.0 Conclusion

A mathematical model based on the Fixed point theory fractional derivative is introduced in the study to study transmission and ways to manage pneumonia. With great importance placed on fractional modeling, we analyzed the fractional pneumonia model in theory, aiming to understand if the answers are unique and stable, along with looking into factors that can influence the spread of the disease in the society. When solving using numbers, Fixed point theory fractional approach was chosen. Simulations proved that the level of disease incidence depends on the chosen model parameters and on fractional orders of the Caputo operator. Besides that, we looked at the results obtained by adjusting important parameters, including the infection in humans and the rate of vaccination among people free of the disease. The analysis proves that by increasing the number of pneumonia treatments and vaccines, the number of cases in the population can be lowered. Other research can look into adopting the methods in [20] to address nonlinear partial differential equations and find analytical solutions.

Data Availability

All the data involved in this study are well cited and given proper references.

Conflict of interest

The authors declared no conflict of interest

Table of Parameters Used

Parameters		**Sources**

Λ_h	0.0413	Estimated
Λ_B	0.0051	Estimated
μ_h	0.0002	[14]
μ_B	0.0001	Estimated
θ_1	0.498	[13]
α_1	0.0238	[16]
σ	0.04	Estimated
β_1	0.0476	Estimated
β_2	0.0275	[15]
δ_1	0.00274	[13]
δ_3	0.001	[13]
ϕ_1	0.33	Estimated
τ_2	0.361	Estimated
τ_1	0.524	Estimated

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