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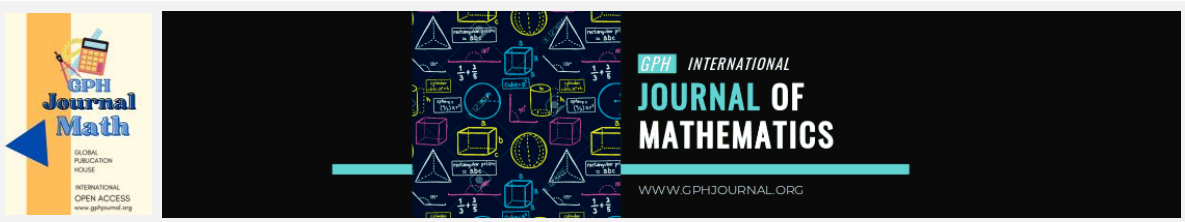


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# Numerical Solution of fractional order Typhoid Fever and HIV/AIDS co-infection Model Via The Generalized Fractional Adams-Bashforth-Moulton Approach

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## Abstract

We investigate the epidemiology of typhoid fever and HIV/AIDs co-infection using a fractional-order model, to understand how treatment affects the transmission of these diseases. In this study we establish situations where solutions exist and are unique and it looks at the stability of the endemic equilibrium by using the Lyapunov function. Applying the fractional Adams–Bashforth–Moulton method, numerically simulations show how the disease is controlled and spread by the chosen parameters. Numerical simulations, suggest that when people have more contacts and treatment is less effective, there is a higher chance of typhoid fever and HIV/AIDs co-infection. Optimizing how treatments are given can greatly limit the spread of the infection in the human population.

## *Keywords and phrases:*

HIV/AIDs, Fractional, Adam-Bashforth-Moulton, Transmission

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## Introduction

To fully understand the effects of infectious diseases, we need a global effort looking at how micro-host, host-host and current social, economic and population factors relate. Severe medical issues called Typhoid fever are caused by infections with the Salmonella typhoid bacteria. WHO evidence uncovers that unclean food and water can help bacteria spread and disease develop when hygiene is not followed. Because the first outbreak of typhoid started in developing countries, typhoid fever now affects many different nations [1]. According to the past figures, Typhoid fever leads to 16 million new cases and around 600,000 deaths each year. There are 198 cases of typhoid fever reported each year for every 100,000 people in the Mekong Delta and 980 new cases per 100,000 people in Delhi, India [1]. Researchers from the 1960s found and shared four methods by which typhoid could get into the body [2].

Having proper sanitation and clean water supplies cuts down the chances of infecting others and makes fewer people sick [3]. Studying the spread of disease and devising prevention plans often requires using mathematical modeling. Using math in disease transmission analysis helps people find key factors and see how diseases spread through the population. Many scientific studies have looked into disease spread through different biological models. Using the right assumptions and parameters, ODEs work very well to represent the different models. Understanding the pattern of typhoid fever spread among vaccinated people was made possible by scientists through their mathematical simulations [4].

Mathematical models are important for investigating the basic biological processes involved in disease spread for public health issues. A lot of research literature on Typhoid fever transmission has been produced by mathematical epidemiology. Scientists work on understanding Typhoid fever and other infectious diseases by modelling them with integer order differential equations [3–10].

Being infected with the Human Immunodeficiency Virus (HIV) may result in Acquired Immunodeficiency Syndrome (AIDS) which reduces humen's immunity and allows fatal infections to enter the body [11]. HIV attacks and disrupts the CD4-T cells of the body which are necessary for the immune system to function. Signs of HIV infection are chills, ongoing diarrhea, tiredness, temperature increases, severe weight loss, night sweats, a sore throat, mouth ulcers and aches in the muscles. Blood, semen, vaginal fluids and breast milk are all ways HIV is transmitted [11]. Because of antiretroviral therapy, people with HIV are living longer, but a cure or vaccine is not yet possible.

Medical reports documented the country's first three cases of AIDS in 1985 and the official declaration of AIDS in Nigeria came from Lagos in 1986. A 13-year-old sex worker living in another West African country was the first Nigerian person known to have AIDS, diagnosed in 1985. When AIDS cases were announced, the majority believed the disease hit primarily homosexual groups in America which led to widespread panic. Nigerians rejected truths about AIDS, concluding it was a way for Americans to curb sexual activity, giving rise to the "AIDS Initiative is an Effort by America to Discourage Sex" (AAIDS).This doctoral dissertation, Modeling HIV/AIDS Epidemic in Nigeria (Eze, 2009), thoroughly investigated people's misunderstanding of how the disease is spread and its effects worldwide.

At the beginning, HIV/AIDS had little effect on Nigeria, but it eventually turned into a big health problem. Year 2014 saw 220,000 new HIV cases and older adults had the highest number of HIV infections in Nigeria. In 2016, the total number of children under 15 with HIV infections was 58,000 and it is estimated that most of these children got HIV from

mothers who have the virus. In 2014, 1.6 million children worldwide had no parents and 747000 people began ART that year [11]. Among all nations, Nigeria has the second highest number of HIV cases (3.2 million), but their HIV infection rate continues to hover around 3.4% with a population of 200 million.

Analyzing infectious diseases, both individually and in combination like typhoid fever and HIV/AIDs, is greatly supported by mathematical modeling. They help researchers discover what causes epidemics and help them design appropriate control measures. Many existing models are useful, but they tend not to consider how past experiences or long-term factors play a role in biology. For this reason, people have started to rely on fractional-order models more often. Bringing in properties from outside the usual assumptions, it now covers memory effects and anomalous diffusion in the spread of diseases [11].

Fractional differential equations (FDEs) build on the standard integer-order approach and can describe more complicated systems. The study uses a fractional-order model to model the spread of typhoid fever and HIV/AIDs co-infection and it also considers ways to control the diseases through treatment and prevention. Because of the memory effect, the model can show how typhoid fever and HIV/AIDs are more likely to be passed together. Various scenarios are tested in simulations, with the purpose of finding out the best approaches to reduce the incidence of both typhoid fever and HIV/AIDs and support long-term infection control.

Diseases like typhoid fever and co-infections of HIV/AIDs can be modeled better using fractional derivatives because these properties consider memory and family influences in biology. They help explore the development of infections over a period and the ways that a person's earlier infection and remedies affect how others may catch the infection. With this effective glance, more progressive and effective measures can be made for managing problems such as drug resistance, being infected again and insufficient healthcare resources.

[12] pointed out that progress in fractional calculus has attracted interest for depicting how different systems change over time. Fractional-order models are different from classical integer-order ones because they take into account the whole system and its memory. Such models not only reflect real life better but also help to address an outbreak which is why they are useful for understanding and addressing diseases like typhoid fever and HIV/AIDs co-infection. In biological applications, fractional derivatives such as the Caputo and Riemann-Liouville derivatives with singular kernels are widely utilized.

Moreover, operators such as the Mittag-Leffler and Atangana-Baleanu which are not used for a single variable, are more widely used now. [12] suggested using fractional-order Sterile Insect Technology (SIT) to fight Zika, solving the model using infinite series that converge to exact values using the Laplace Asymptotic Dominance Method (LADM). [13] looked into spread of Lassa fever by using a fractional-order model which used a power-law fractional derivative to explore how vaccination and treatment reduced its spread. [14] examined how Nigeria dealt with COVID-19 using Caputo derivative and the LADM model, reporting that recovery became higher in the integer-order models, as a result of people being vaccinated and treated. [15] designed a fractional-order model using the Caputo derivative to characterize soil-transmitted helminth infections. Applying the LADM, they revealed that the model’s infinite series correctly approached ideal values which made it more flexible than norms that only involve integer-order approximations. [16] used fractional mathematics to build a model of how hepatitis C spreads, using the Adams-Bashforth-Moulton technique.

Researchers found that lowering the rate of contacts and improving treatment greatly decreased the spread of the disease and the model performed better than the older models. [17] introduced a fractional order system to understand HIV/AIDS transmission, alongside the Adams-Bashforth-Moulton method. It was revealed that bringing down disease transmissions and improving ways to treat used were effective in managing the disease, as well as how useful fractional models are compared to the old ones. The model in [18], again based on the Adams-Bashforth-Moulton method, was a fractional order for tracking Diphtheria transmission. They found that when contact rates were lowered, the disease rates went down, as did the ineffectiveness of treatment which helps prove the growing importance of these fractional-order techniques. [19] used the model of fractional derivative to model and predict the interactions between COVID-19 and HIV. [20] looked at how hepatitis C and COVID-19 infections can happen together. They brought together recent mathematical modeling studies, focusing on important techniques, key findings and places where researchers may need to examine more.

One advantage of fractional-order models is they are very flexible and are able to include effects that are not found locally. Fractional derivatives differ from classical derivatives by giving a better fit to real-world scenarios and more adaptability. They rely on interactions that are not limited to local areas which is often missed in standard models and this allows them to describe things like memory effects which is not possible with integer-order derivatives. Because of these properties, other types of equations, like fractional differential equations, are now being used by researchers. [21] as seen in [22] dealt with Volterra integral equations that have degenerate kernels by employing a combination of the Laplace transform and Adomian Decomposition Method. Because of this approach, attention has been brought to how fuzzy analytical dynamic equations are evolving. Ulam stability and the existence of solutions were explored for a three-point boundary value problem in [23]. They used non-linear fractional approaches from classical mathematics to investigate the stability of the problem which proved insightful.

The objective of the paper is to:

- Find out what conditions must exist to make sure the proposed model has a unique and valid solution.
- Study whether the endemic equilibrium point is stable by applying the Lyapunov stable function method.
- Derive numerical answers using the fractional Adams–Bashforth–Moulton method.
- Run numerical experiments to see how the model reacts.

No paper has focus on modeling and analyzing HIV/AIDS and typhoid fever co-infection using the combination of fractional calculus and the Adams–Bashforth–Moulton method to the best of the researcher's knowledge.

The paper is organized so that Section 2 sets out the mathematical model, Section 3 examines the model analytically, Section 4 shows the numerical results for the fractional-order approach and Section 5 concludes with an abstract and final viewpoint.

## Preliminary

This section covers fundamental facts and major ideas from fractional calculus. The analysis includes both the right and left fractional Caputo derivatives, similar to what was done in [25,

26]. The paper also points out the uses of fractional calculus in solving real challenges that appear in physics, engineering, bio-mathematics and other scientific sectors.

**Definition 1**: Let $f \in \Lambda^{\infty}(R)$ then the left and right Caputo fractional derivative of the function
$f$ is given by

$$ {}^{C}D_{t}^{\gamma} f(t) = \left( t^{0}D_{t}^{-(n-\gamma)} \left( \frac{d}{dt} \right)^{n} f(t) \right) $$

$$ {}^{C}D_{t}^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_{0}^{t} \left( (t-\lambda)^{n-\gamma-1} f^{n}(\lambda) \right) d\lambda \quad (1) $$

The same way

$$ {}^{C}D_{t}^{\gamma} f(t) = \left( D_{T}^{-(n-\gamma)} \left( \frac{-d}{dt} \right)^{n} \right) f(t) $$

$$ {}^{C}D_{T}^{\gamma} f(t) = \frac{(-1)^{n}}{\Gamma(n-\gamma)} \int_{t}^{T} (\lambda - t)^{n-\gamma-1} f^{n}(\lambda) d\lambda $$

Definition 2: The generalized Mittag-Leffler function $E_{\alpha,\beta}(x)$ for $x \in R$ is given by

$$ E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^{n}}{\Gamma(\alpha n + \psi)}, \ \alpha, \psi > 0 \ (2) $$

which can also be represented as

$$ E_{\alpha,\psi}(x) = xE_{\alpha,\alpha+\psi(x)} + \frac{1}{\Gamma(\psi)} \quad (3) $$

$$ E_{\alpha,\psi}(x) = L\left[ t^{\psi-1} E_{\alpha,\psi}\left( \pm \omega t^{\alpha} \right) \right] = \frac{S^{\alpha-\psi}}{S^{\alpha} \pm \omega} \quad (4) $$

Proposition 1.1.

Let $f \in \Lambda^{\infty}(R) \cap C(R)$ and $\alpha \in R, n-1 < \alpha < n,$

Therefore, the conditions given below holds:

1. ${}^{C}_{t_0}D_{t}^{\gamma} I^{\gamma} f(t) = f(t)$

2. ${}^{C}_{t_0}D_{t}^{\gamma} I^{\gamma} f(t) = f(t) - \sum_{k-0}^{n-k} \frac{t^{k}}{K!} f^{k}(t_0)$

**Model Flow Chart**

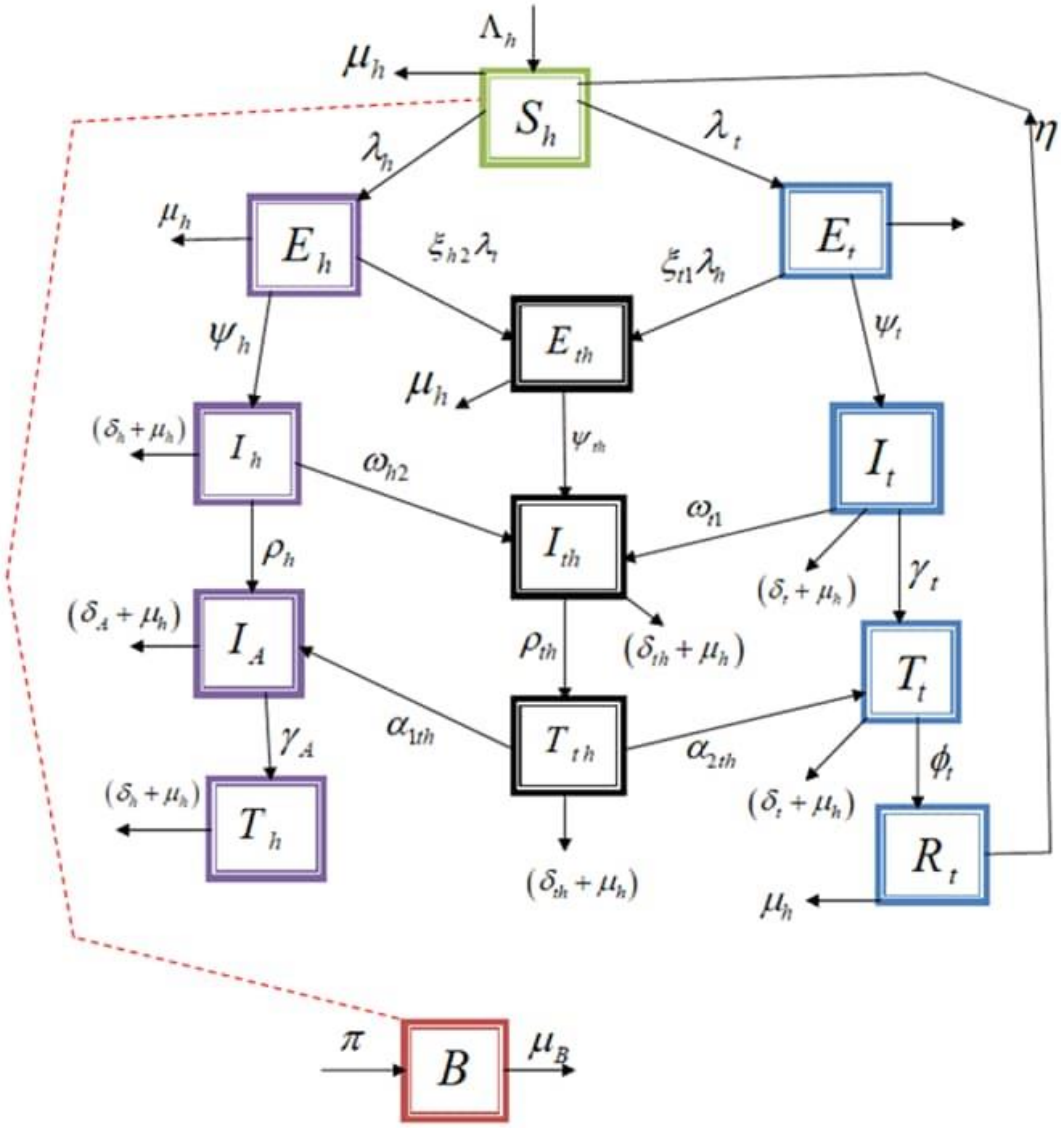


Fig.1: Model Flow Diagram of Typhoid fever and HIV/AIDS co-infection

## 5.3 Model Equation

$$\frac{dS_h}{dt} = \Lambda_h + \eta R_t - \lambda_h S_h - \lambda_t S_h - \mu_h S_h,$$

$$\frac{dE_t}{dt} = \lambda_t S_h - \left(\lambda_h \xi_{t1} + \psi_t + \mu_h\right) E_t,$$

$$\frac{dE_h}{dt} = \lambda_h S_h - \left(\lambda_t \xi_{t2} + \psi_h + \mu_h\right) E_h,$$

$$\frac{dE_{th}}{dt} = \lambda_t \xi_{h2} E_h + \lambda_h \xi_{t1} E_t - \left(\psi_{th} + \mu_h\right) E_{th},$$

$$\frac{dI_h}{dt} = \psi_{h1}E_h - \left(\omega_{h2} + \rho_h + \delta_h + \mu_h\right)I_h,$$

$$\frac{dI_A}{dt} = \rho_h I_h - \left(\gamma_A + \delta_A + \mu_h\right)I_A,$$

$$\frac{dI_t}{dt} = \psi_t E_t - \left(\gamma_t + \delta_t + \mu_h\right)I_t,$$

$$\frac{dI_{th}}{dt} = \psi_{th}E_{th} + \psi_{h2}I_h + \psi_{t1}I_t - \left(\rho_{th} + \delta_{th} + \mu_h\right)I_{th},$$

$$\frac{dT_h}{dt} = \gamma_A I_A + \alpha_{1th}T_{th} - \left(\delta_h + \mu_h\right)T_h,$$

$$\frac{dT_t}{dt} = \gamma_t I_t + \alpha_{2th}T_{th} - \left(\phi_t + \delta_t + \mu_h\right)T_t,$$

$$\frac{dT_{th}}{dt} = \rho_{th}I_{th} - \left(\alpha_{1th} + \alpha_{2th} + \delta_{th} + \mu_h\right)T_{th},$$

$$\frac{dR_t}{dt} = \phi_t T_t - \left(\eta + \mu_h\right)R_t,$$

$$\frac{dB}{dt} = \pi - \mu_B B.$$

Where $\lambda_t = \frac{\left(\beta_{t1}I_t + \beta_{t2}I_{th} + \beta_{t3}I_{tA} + \beta_{t4}B\right)}{N_h}$ shows the infection rate that results from the way typhoid is spread. $\lambda_h = \frac{\left(\beta_1 I_h + \beta_2 I_A + \beta_3 T_h + \beta_4 I_{tA} + \beta_5 T_{th} + \beta_6 I_{th}\right)}{N_h}$ shows the infection rate that results from the way HIV/AIDS is spread.

**Table 1: Descriptions of Variables and Parameters**

| Variables | Description |
|---|---|
| $S_h(t)$ | Humans who are susceptible |
| $E_t(t)$ | Humans exposed exclusively to typhoid fever |

| | |
|---|---|
| $E_h(t)$ | Humans exposed exclusively to HIV |
| $E_{th}(t)$ | Humans exposed solely to typhoid and HIV |
| $I_t(t)$ | Humans infected solely with typhoid |
| $I_h(t)$ | Humans infected solely with HIV |
| $I_A(t)$ | Humans infected exclusively with HIV/AIDS |
| $I_{th}(t)$ | Humans infected solely with typhoid and HIV |
| $I_{tA}(t)$ | People have tested positive for AIDS and typhoid at the same time in this study |
| $T_t(t)$ | Treated class of typhoid patients only |
| $T_h(t)$ | Treated class of HIV patients only |
| $T_{th}(t)$ | Treated class of typhoid and HIV patients |
| $R_t(t)$ | Recovered typhoid patients' class |
| $S_{vd}(t)$ | Susceptible mosquitoes |
| $B$ | Bacteria causing typhoid |
| **Parameters** | Description |
| $\Lambda_h$ | The rate at which humans are recruited |
| $\mu_h, \mu_B$ | The natural mortality rate of humans, and typhoid bacteria. |
| $\varepsilon_{t2}$ | A modification parameter that adjusts for the reduced co-exposure of typhoid and HIV |
| $\varepsilon_{h2}$ | A modification factor that reduces the co-exposure of HIV and typhoid |
| $\psi_t, \psi_h$ | The number of people who move from typhoid, and HIV exposure to infection status of typhoid, and HIV, respectively. |
| $\omega_{t2}$ | The path from one disease state of just typhoid fever to the joint condition of typhoid and HIV infection. |
| $\omega_{h2}$ | The progression rate from individuals infected with only HIV to the co-infected class of typhoid and HIV |

| | |
|---|---|
| $\omega_{h2}$ | The speed at which individuals with HIV alone transition to being co-infected with both typhoid and HIV. |
| $\theta_{th}$ | The progression rate from the co-infected class of typhoid and HIV only to the co-infected class of typhoid, and HIV. |
| $\rho_h$ | How fast HIV progresses from its first infection stage to full-blown AIDS symptoms. |
| $\rho_{th}$ | The progression rate from the co-infected typhoid-HIV class to the typhoid-HIV/AIDS class. |
| $\gamma_t$ | Treatment rate of infected typhoid only class. |
| $\gamma_h$ | The treatment rate for the infected HIV-only class |
| $\gamma_A$ | The treatment rate for the infected HIV/AIDS-only class |
| $\gamma_{th}$ | The co-infected typhoid and HIV population receives different levels of treatment compared to patients who have only HIV. |
| $\gamma_{tA}$ | Treatment rate of co-infected typhoid fever and HIV/AIDS only class. |
| $\phi_t$ | Patients treated for typhoid fever will recover completely. |
| $\mu_h, \mu_B$ | The natural mortality rate of humans, dengue fever mosquitoes, and bacteria. |
| $\delta_t, \delta_h, \delta_A, \delta_{th},$ | The mortality rate associated with diseases in individuals infected with typhoid, HIV, or HIV/AIDS, including those co-infected with typhoid-HIV, typhoid-HIV/AIDS. |
| $\eta$ | Rate of re-infection of recovered typhoid fever humans. |
| $\beta_t$ | Contact rate of susceptible humans with typhoid infected classes. |

## 5.4 Fractional order mathematical Model

In this section we transform the integer order model to fractional order as follows:

$$^{c}D_t^{\gamma} S_h = \Lambda_h + \eta R_t - \lambda_h S_h - \lambda_t S_h - \mu_h S_h,$$

$$
\begin{aligned}
{}^{c}D_t^{\gamma} E_t &= \lambda_t S_h - \left(\lambda_h \xi_{t1} + \psi_t + \mu_h\right) E_t, \\
{}^{c}D_t^{\gamma} E_h &= \lambda_h S_h - \left(\lambda_t \xi_{t2} + \psi_h + \mu_h\right) E_h, \\
{}^{c}D_t^{\gamma} E_{th} &= \lambda_t \xi_{h2} E_h + \lambda_h \xi_{t1} E_t - \left(\psi_{th} + \mu_h\right) E_{th}, \\
{}^{c}D_t^{\gamma} I_h &= \psi_{h1} E_h - \left(\omega_{h2} + \rho_h + \delta_h + \mu_h\right) I_h, \\
{}^{c}D_t^{\gamma} I_A &= \rho_h I_h - \left(\gamma_A + \delta_A + \mu_h\right) I_A, \\
{}^{c}D_t^{\gamma} I_t &= \psi_t E_t - \left(\gamma_t + \delta_t + \mu_h\right) I_t, \\
{}^{c}D_t^{\gamma} I_{th} &= \psi_{th} E_{th} + \psi_{h2} I_h + \psi_{t1} I_t - \left(\rho_{th} + \delta_{th} + \mu_h\right) I_{th}, \\
{}^{c}D_t^{\gamma} T_h &= \gamma_A I_A + \alpha_{1th} T_{th} - \left(\delta_h + \mu_h\right) T_h, \\
{}^{c}D_t^{\gamma} T_t &= \gamma_t I_t + \alpha_{2th} T_{th} - \left(\phi_t + \delta_t + \mu_h\right) T_t, \\
{}^{c}D_t^{\gamma} T_{th} &= \rho_{th} I_{th} - \left(\alpha_{1th} + \alpha_{2th} + \delta_{th} + \mu_h\right) T_{th}, \\
{}^{c}D_t^{\gamma} R_t &= \phi_t T_t - \left(\eta + \mu_h\right) R_t, \\
{}^{c}D_t^{\gamma} B &= \pi - \mu_B B.
\end{aligned}
$$

Subject to the initial condition s;

$$
\begin{aligned}
&S_h(0) = S_{h0}, \mathrm{E}_h(0) = \mathrm{E}_{h0}, \mathrm{E}_t(0) = \mathrm{E}_{t0}, \mathrm{E}_{th}(0) = \mathrm{E}_{th0}, \mathrm{I}_h(0) = \mathrm{I}_{h0}, \\
&\mathrm{I}_A(0) = \mathrm{I}_{A0}, \mathrm{I}_t(0) = \mathrm{I}_{t0}, \mathrm{I}_{th}(0) = \mathrm{I}_{th0}, T_h(0) = T_{h0}, T_t(0) = T_{t0}, \\
&T_{th}(0) = T_{th0}, \mathrm{R}_t(0) = R_{t0}.
\end{aligned}
$$

## 5.5 Model Analysis

## 5.6 Positivity of Model solution

We considered the non-negativity of the initial values

$$\lim Sup.N_h(t) \leq \frac{\Lambda_h}{\mu_h},$$

Secondly, If $\lim Sup.N_{h0}(t) \le \frac{\Lambda_h}{\mu_h}$, then our model feasible domain is given by:

$$\Omega = \left\{ \begin{array}{l} \left(S_h, \mathrm{E}_h, \mathrm{E}_t, \mathrm{E}_{th}, \mathrm{I}_h, \mathrm{I}_A, \mathrm{I}_t, \mathrm{I}_{th}, T_h, T_t, T_{th}\, \mathrm{R}_t\right) \subset \mathrm{R}^{12} \\ : \left(S_h + \mathrm{E}_h + \mathrm{E}_t + \mathrm{E}_{th} + \mathrm{I}_h + \mathrm{I}_A + \mathrm{I}_t + \mathrm{I}_{th} + T_h + T_t + T_{th} + \mathrm{R}_t\right) \le \frac{\Lambda_h}{\mu_h}, \end{array} \right\},$$

so that

$$\Omega = \Omega_{ht} \subset R_+^{13},$$

hence $\Omega$ is positively invariant.

If $\left(S_{h0}, \mathrm{E}_{h0}, \mathrm{E}_{t0}, \mathrm{E}_{th0}, \mathrm{I}_{h0}, \mathrm{I}_{A0}, \mathrm{I}_{t0}, \mathrm{I}_{th0}, T_{h0}, T_{t0}, T_{th0}\, \mathrm{R}_{t0}\right)$ For model (6), if the coefficients are non-negative, then the solution will also be non-negative for all times $t > 0$. Selecting the first equation from Eq. (6), we can say that:.

$$^{c}D_t^{\gamma} S_h = \Lambda_h + \eta R_t - \lambda_h S_h - \lambda_t S_h - \mu_h S_h,$$

$$^{c}D_t^{\gamma} S_h = \Lambda_h + \eta R_t - \left(\lambda_h + \lambda_t + \mu_h\right) S_h,$$

$$^{c}D_t^{\gamma} S_h + \left(\lambda_h + \lambda_t + \mu_h\right) S_h = \Lambda_h + \eta R_t,$$

But $\Lambda_h + \eta R_t \ge 0$, then, $^{c}D_t^{\gamma} S_h + \left(\lambda_h + \lambda_t + \mu_h\right) S_h \ge 0$.

Using the Laplace transform we have;

$$L\left[^{c}D_t^{\gamma} S_h\right] + L\left[\left(\lambda_h + \lambda_t + \mu_h\right) S_h\right] \ge 0$$

$$S_h^{\gamma} S_h(s) - S_h^{\gamma-1} S_h(0) + \left(\left(\lambda_h + \lambda_t + \mu_h\right)\right) S_h(s) \ge 0,$$

$$S_h^{\gamma}(s) \ge \frac{S_h^{\gamma-1}}{S_h^{\gamma} + \left(\lambda_h + \lambda_t + \mu_h\right)} S_h(0). \qquad (8)$$

By taking the inverse Laplace transform, we obtained;

$$S_h(t) \ge E_{r,1}\left(\lambda_h + \lambda_t + \mu_h t^{\gamma}\right) \mathrm{S}_{h0}. \qquad (9)$$

Now since the term on the right hand side of Eq. (9) is positive, we conclude that $S_h \ge 0$ for $t \ge 0$. In the same way, we also have that; $\left(S_h \ge 0, \mathrm{E}_h \ge 0, \mathrm{E}_t \ge 0, \mathrm{E}_{th} \ge 0, \mathrm{I}_h \ge 0, \mathrm{I}_A \ge 0, \mathrm{I}_t \ge 0, \mathrm{I}_{th} \ge 0, T_h \ge 0, T_t \ge 0, T_{th} \ge 0\, \mathrm{R}_t \ge 0\right)$. that are positives, therefore, the solution will remain in $R_+^{12}$ for all $t \ge 0$ with positive initial conditions.

### 5.7 Boundedness of fractional model solution.

The total population of individuals from our model is given by ;

$$N_h(t) = S_h(t) + \mathrm{E}_h(t) + \mathrm{E}_t(t) + \mathrm{E}_{th}(t) + \mathrm{I}_h(t) + \mathrm{I}_A(t) + \mathrm{I}_t(t)$$
$$+ \mathrm{I}_{th}(t) + T_h(t) + T_t(t) + T_{th}(t) + \mathrm{R}_t(t).$$

So from our fractional model (6), we now obtain

$${}^{c}D_t^{\gamma} N_h(t) = {}^{c}D_t^{\gamma} S_h(t) + {}^{c}D_t^{\gamma} \mathrm{E}_h(t) + {}^{c}D_t^{\gamma} \mathrm{E}_t(t) + {}^{c}D_t^{\gamma} \mathrm{E}_{th}(t) + {}^{c}D_t^{\gamma} \mathrm{I}_h(t) + {}^{c}D_t^{\gamma} \mathrm{I}_A(t)$$
$$+ {}^{c}D_t^{\gamma} \mathrm{I}_t(t) + {}^{c}D_t^{\gamma} \mathrm{I}_{th}(t) + {}^{c}D_t^{\gamma} T_h(t) + {}^{c}D_t^{\gamma} T_t(t) + {}^{c}D_t^{\gamma} T_{th}(t) + {}^{c}D_t^{\gamma} \mathrm{R}_t(t).$$

$${}^{c}D_t^{\gamma} N_h(t) = \Lambda - \mu_h N_h(t) \qquad (10)$$

Taking the Laplace transformation of (10) we obtained;

$$L\left[{}^{c}D_t^{\gamma} N_h(t)\right] \le L\left[\Lambda_h - \mu_h N_h(t)\right],$$

$$S_h^{\gamma} N_h(\mathrm{s}) - S_h^{\gamma-1} N_h(0) + \mu_h N_h(s) \le \frac{\Lambda_h}{\mu_h},$$

$$N_h(\mathrm{s}) \le \frac{S_h^{\gamma-1}}{\left(S_h^{\gamma} + \mu\right)} N_h(0) + \frac{\Lambda_h}{S_h\left(S_h^{\gamma} + \mu_h\right)} \qquad (11)$$

By taking the inverse Laplace transform of Eq. (11) we obtained;

$$N_h(\mathrm{t}) \le E_{r,1}\left(-\mu_h t^{\gamma}\right) N_h(0) + \Lambda_h E_{r,\mathrm{r}+1}\left(-\mu_h t^{\gamma}\right) \qquad (12)$$

At $t \to \infty$, the limit of Eq. (12) becomes

$$\lim_{t\to\infty} Sup N_h(t) \le \frac{\Lambda_h}{\mu_h}.$$

This means that, if $N_{h0} \le \frac{\Lambda_h}{\mu_h}$ then $N_h(t) \le \frac{\Lambda_h}{\mu_h}$ which implies that, $N_h(t)$ is bounded.

We now conclude that, this region $\Omega = \Omega_{ht}$, is well posed and equally feasible epidemiologically.

### 5.8 Existence and uniqueness of our model solution

Let the real non-negative be P, we consider $Q = [0, P]$.

The set of all continuous function that is defined on M is represented by $N_e^0(Q)$ with norm as;

$$\|X\| = Sup\left\{\left|X(t)\right|, \mathrm{t} \in \mathrm{Q}\right\}.$$

Starting with the first model in (6) at initial conditions stated in (8), then the problem becomes an initial value problem (IVP), as stated in (13).

$$ {}^{c}D_t^{\gamma}\left(t\right) = Z\left(t, X\left(t\right)\right), 0 < t < P < \infty,$$

$$X\left(0\right) = X_0.$$

Where $Y\left(t\right) = \left(S_h, \mathrm{E}_h, \mathrm{E}_t, \mathrm{E}_{th}, \mathrm{I}_h, \mathrm{I}_A, \mathrm{I}_t, \mathrm{I}_{th}, T_h, T_t, T_{th}\, \mathrm{R}_t\right)$ represents the classes and Z be a continuous function defined as follows;

$$\begin{pmatrix} Z_1\left(t, S_h\left(t\right)\right) \\ Z_2\left(t, \mathrm{E}_h\left(t\right)\right) \\ Z_3\left(t, \mathrm{E}_t\left(t\right)\right) \\ Z_4\left(t, \mathrm{E}_{th}\left(t\right)\right) \\ Z_5\left(t, \mathrm{I}_h\left(t\right)\right) \\ Z_6\left(t, \mathrm{I}_A\left(t\right)\right) \\ Z_7\left(t, \mathrm{I}_t\left(t\right)\right) \\ Z_8\left(t, \mathrm{I}_{th}\left(t\right)\right) \\ Z_9\left(t, \mathrm{T}_h\left(t\right)\right) \\ Z_{10}\left(t, \mathrm{T}_t\left(t\right)\right) \\ Z_{11}\left(t, \mathrm{T}_{th}\left(t\right)\right) \\ Z_{12}\left(t, \mathrm{T}_t\left(t\right)\right) \\ Z_{13}\left(t, B\left(t\right)\right) \end{pmatrix} = \begin{pmatrix} \Lambda_h + \eta R_t - \lambda_h S_h - \lambda_t S_h - \mu_h S_h \\ \lambda_h S_h - \left(\lambda_t \xi_{t2} + \psi_h + \mu_h\right) E_h \\ \lambda_t S_h - \left(\lambda_h \xi_{t1} + \psi_t + \mu_h\right) E_t \\ \lambda_t \xi_{h2} E_h + \lambda_h \xi_{t1} E_t - \left(\psi_{th} + \mu_h\right) E_{th} \\ \psi_{h1} E_h - \left(\omega_{h2} + \rho_h + \delta_h + \mu_h\right) I_h \\ \rho_h I_h - \left(\gamma_A + \delta_A + \mu_h\right) I_A \\ \psi_t E_t - \left(\gamma_t + \delta_t + \mu_h\right) I_t \\ \psi_{th} E_{th} + \psi_{h2} I_h + \psi_{t1} I_t - \left(\rho_{th} + \delta_{th} + \mu_h\right) I_{th} \\ \gamma_A I_A + \alpha_{1th} T_{th} - \left(\delta_h + \mu_h\right) T_h \\ \gamma_t I_t + \alpha_{2th} T_{th} - \left(\phi_t + \delta_t + \mu_h\right) T_t \\ \rho_{th} I_{th} - \left(\alpha_{1th} + \alpha_{2th} + \delta_{th} + \mu_h\right) T_{th} \\ \phi_t T_t - \left(\eta + \mu_h\right) R_t \\ \pi - \mu_B B \end{pmatrix} \tag{14}$$

Using proposition (2.1), we have that,

$$\begin{aligned} S_h\left(t\right) &= S_{h0} + I_t^{\gamma}\left[\Lambda_h + \eta R_t - \lambda_h S_h - \lambda_t S_h - \mu_h S_h\right], \\ E_h\left(t\right) &= E_{h0} + I_t^{\gamma}\left[\lambda_h S_h - \left(\lambda_t \xi_{t2} + \psi_h + \mu_h\right) E_h\right], \\ E_t\left(t\right) &= E_{t0} + I_t^{\gamma}\left[\lambda_t S_h - \left(\lambda_h \xi_{t1} + \psi_t + \mu_h\right) E_t\right], \\ E_{th}\left(t\right) &= E_{th0} + I_t^{\gamma}\left[\lambda_t \xi_{h2} E_h + \lambda_h \xi_{t1} E_t - \left(\psi_{th} + \mu_h\right) E_{th}\right], \\ I_h\left(t\right) &= I_{h0} + I_t^{\gamma}\left[\psi_{h1} E_h - \left(\omega_{h2} + \rho_h + \delta_h + \mu_h\right) I_h\right], \\ I_A\left(t\right) &= I_{A0} + I_t^{\gamma}\left[\rho_h I_h - \left(\gamma_A + \delta_A + \mu_h\right) I_A\right], \\ I_t\left(t\right) &= I_{t0} + I_t^{\gamma}\left[\psi_t E_t - \left(\gamma_t + \delta_t + \mu_h\right) I_t\right], \end{aligned} \tag{15}$$

$$I_{th}(t) = I_{th} + I_t^{\gamma}\left[\psi_{th}E_{th} + \psi_{h2}I_h + \psi_{t1}I_t - (\rho_{th} + \delta_{th} + \mu_h)I_{th}\right],$$

$$T_h(t) = T_{h0} + I_t^{\gamma}\left[\gamma_A I_A + \alpha_{1th}T_{th} - (\delta_h + \mu_h)T_h\right],$$

$$T_t(t) = T_{t0} + I_t^{\gamma}\left[\gamma_t I_t + \alpha_{2th}T_{th} - (\phi_t + \delta_t + \mu_h)T_t\right],$$

$$T_{th}(t) = T_{th0} + I_t^{\gamma}\left[\rho_{th}I_{th} - (\alpha_{1th} + \alpha_{2th} + \delta_{th} + \mu_h)T_{th}\right],$$

$$R_t(t) = R_{t0} + I_t^{\gamma}\left[\phi_t T_t - (\eta + \mu_h)R_t\right],$$

$$B(t) = B_0 + I_t^{\gamma}\left[\pi - \mu_B B\right].$$

We obtain the Picard iteration of (15) as follows;

$$S_{hn}(t) = S_{h0} + \frac{1}{\Gamma(\gamma)}\int_0^t (t-\lambda)^{\gamma-1} Z_1\left(\lambda, S_{hn-1}(\lambda)\right)\mathrm{d}\lambda,$$

$$E_h(t) = E_{h0} + \frac{1}{\Gamma(\gamma)}\int_0^t (t-\lambda)^{\gamma-1} Z_2\left(\lambda, \mathrm{E}_{\mathrm{h}(n-1)}(\lambda)\right)\mathrm{d}\lambda,$$

$$E_{tn}(t) = E_{t0} + \frac{1}{\Gamma(\gamma)}\int_0^t (t-\lambda)^{\gamma-1} Z_3\left(\lambda, \mathrm{E}_{\mathrm{t}(n-1)}(\lambda)\right)\mathrm{d}\lambda,$$

$$E_{thn}(t) = E_{th0} + \frac{1}{\Gamma(\gamma)}\int_0^t (t-\lambda)^{\gamma-1} Z_4\left(\lambda, \mathrm{E}_{\mathrm{th}(n-1)}(\lambda)\right)\mathrm{d}\lambda,$$

$$I_{hn}(t) = I_{h0} + \frac{1}{\Gamma(\gamma)}\int_0^t (t-\lambda)^{\gamma-1} Z_6\left(\lambda, I_{h(n-1)}(\lambda)\right)\mathrm{d}\lambda,$$

$$I_{An}(t) = I_{A0} + \frac{1}{\Gamma(\gamma)}\int_0^t (t-\lambda)^{\gamma-1} Z_7\left(\lambda, I_{A(n-1)}(\lambda)\right)\mathrm{d}\lambda, \quad (16)$$

$$I_{tn}(t) = I_{t0} + \frac{1}{\Gamma(\gamma)}\int_0^t (t-\lambda)^{\gamma-1} Z_8\left(\lambda, I_{t(n-1)}(\lambda)\right)\mathrm{d}\lambda,$$

$$I_{thn}(t) = I_{th0} + \frac{1}{\Gamma(\gamma)}\int_0^t (t-\lambda)^{\gamma-1} Z_9\left(\lambda, I_{th(n-1)}(\lambda)\right)\mathrm{d}\lambda,$$

$$T_{hn}(t) = T_{h0} + \frac{1}{\Gamma(\gamma)}\int_0^t (t-\lambda)^{\gamma-1} Z_{10}\left(\lambda, T_{h(n-1)}(\lambda)\right)\mathrm{d}\lambda,$$

$$T_{tn}(t) = T_{t0} + \frac{1}{\Gamma(\gamma)}\int_0^t (t-\lambda)^{\gamma-1} Z_{11}\left(\lambda, T_{t(n-1)}(\lambda)\right)\mathrm{d}\lambda,$$

$$R_{tn}\left(t\right)=R_{t0}+\frac{1}{\Gamma\left(\gamma\right)}\int_0^t\left(t-\lambda\right)^{\gamma-1}Z_{12}\left(\lambda,R_{t(n-1)}\left(\lambda\right)\right)\mathrm{d}\lambda,$$

Transforming the initial value problem of Eq. (13) we have;

$$X\left(t\right)=X\left(0\right)+\frac{1}{\Gamma\left(\gamma\right)}\int_0^t\left(t-\lambda\right)^{\gamma-1}Z\left(\lambda,X\left(\lambda\right)\right)\mathrm{d}\lambda. \qquad (17)$$

Lemma 1, The Lipchitz condition described from Eq. (14) is satisfied by vector $Z\left(\mathrm{t},X\left(t\right)\right)$ on a set $\left[0,P\right]\times R_+^{12}$ with the Lipchitz constant given as;

$$\beta=\max\begin{pmatrix}\left(\beta_{t1}^*+\beta_{t2}^*+\beta_{t3}^*+\beta_{t4}^*+\beta_{t5}^*\right),\left(\beta_1^*+\beta_2^*+\beta_3^*+\beta_4^*+\beta_5^*+\beta_6^*\right),\left(\xi_{h2}+\psi_h+\mu_h\right),\left(\xi_{h1}+\psi_t+\mu_h\right)\\,\left(\psi_{th}+\mu_h\right),\left(\omega_{h2}+\rho_h+\delta_h+\mu_h\right),\ \left(\gamma_A+\delta_A+\mu_h\right),\ \left(\gamma_t+\delta_t+\mu_h\right),\left(\rho_{th}+\delta_{th}+\mu_h\right)\\\left(\delta_h+\mu_h\right),\left(\phi_t+\delta_t+\mu_h\right),\left(\alpha_{1th}+\alpha_{2th}+\delta_6+\mu_h\right),\left(\eta+\mu_h\right),\left(\mu_h\right)\end{pmatrix}.$$

Proof.

$$\left\|Z_1\left(t,S\right)-Z_1\left(t,S_1\right)\right\|$$

$$=\left\|\Lambda_h+\eta R_t-\left(\lambda_h+\lambda_t+\mu_h\right)S_h-\Lambda_h+\eta R_t-\left(\lambda_h+\lambda_t+\mu_h\right)S_{h1}\right\|$$

$$=\left\|-\left(\lambda_h+\lambda_t+\mu_h\right)\left(S_h-S_{h1}\right)+\mu_h\left(S_h-S_{h1}\right)\right\|\le\left(\lambda_h^*+\lambda_t^*\right)\left\|S_h-S_{h1}\right\|+\mu_h\left\|S_h-S_{h1}\right\|$$

$$\therefore\left\|Z_1\left(t,S_h\right)-Z_1\left(t,S_{h1}\right)\right\|\le\left(\lambda_h^*+\lambda_t^*\right)\left\|S_h-S_{h1}\right\|+\mu_h\left\|S_h-S_{h1}\right\|$$

Similarly we obtained the following;

$$\left\|Z_2\left(t,\mathrm{E}_h\right)-Z_2\left(t,\mathrm{E}_{h1}\right)\right\|\le\left(\psi_h+\mu_h\right)\left\|\mathrm{E}_h-\mathrm{E}_{h1}\right\|,$$

$$\left\|Z_3\left(t,\mathrm{E}_t\right)-Z_3\left(t,\mathrm{E}_{t1}\right)\right\|\le\left(\psi_t+\mu_h\right)\left\|\mathrm{E}_t-\mathrm{E}_{t1}\right\|,$$

$$\left\|Z_4\left(t,\mathrm{E}_{th}\right)-Z_4\left(t,\mathrm{E}_{th1}\right)\right\|\le\left(\psi_{th}+\mu_h\right)\left\|\mathrm{E}_{th}-\mathrm{E}_{th1}\right\|,$$

$$\left\|Z_5\left(t,\mathrm{I}_h\right)-Z_5\left(t,\mathrm{I}_{h1}\right)\right\|\le\left(\omega_{h2}+\rho_h+\delta_h+\mu_h\right)\left\|\mathrm{I}_h-\mathrm{I}_{h1}\right\|,$$

$$\left\|Z_6\left(t,\mathrm{I}_A\right)-Z_6\left(t,\mathrm{I}_{A1}\right)\right\|\le\left(\gamma_A+\delta_A+\mu_h\right)\left\|\mathrm{I}_A-\mathrm{I}_{A1}\right\|,$$

$$\left\|Z_7\left(t,\mathrm{I}_t\right)-Z_7\left(t,\mathrm{I}_{t1}\right)\right\|\le\left(\gamma_t+\delta_t+\mu_h\right)\left\|\mathrm{I}_t-\mathrm{I}_{t1}\right\|, \qquad (18)$$

$$\left\|Z_8\left(t,\mathrm{I}_{th}\right)-Z_8\left(t,\mathrm{I}_{th1}\right)\right\|\le\left(\rho_{th}+\delta_{th}+\mu_h\right)\left\|\mathrm{I}_{th}-\mathrm{I}_{th1}\right\|,$$

$$\left\|Z_9\left(t,\mathrm{T}_h\right)-Z_9\left(t,\mathrm{T}_{h1}\right)\right\|\le\left(\delta_h+\mu_h\right)\left\|\mathrm{T}_h-\mathrm{T}_{h1}\right\|,$$

$$\left\|Z_{10}\left(t,\mathrm{T}_t\right)-Z_{10}\left(t,\mathrm{T}_{t1}\right)\right\|\le\left(\phi_t+\delta_t+\mu_h\right)\left\|\mathrm{T}_t-\mathrm{T}_{t1}\right\|,$$

$$\left\|Z_{11}\left(t,\mathrm{T}_{th}\right)-Z_{11}\left(t,\mathrm{T}_{th1}\right)\right\|\leq\left(\alpha_{1th}+\alpha_{2th}+\delta_{th}+\mu_h\right)\left\|\mathrm{T}_{th}-\mathrm{T}_{th1}\right\|$$

$$\left\|Z_{12}\left(t,R_t\right)-Z_{12}\left(t,R_{t1}\right)\right\|\leq\left(\eta+\mu_h\right)\left\|R_t-R_{t1}\right\|$$

$$\left\|Z_{13}\left(t,B\right)-Z_{13}\left(t,B_1\right)\right\|\leq\left(\mu_h\right)\left\|B-B_1\right\|.$$

Where we obtained

$$\left\|Z\left(t,\mathrm{X}_1\left(t\right)\right)-Z\left(t,\mathrm{X}_2\left(t\right)\right)\right\|\leq\beta\left\|\mathrm{X}_1-\mathrm{X}_2\right\|,$$

$$\beta=\max\begin{pmatrix}\left(\beta_{t1}^*+\beta_{t2}^*+\beta_{t3}^*+\beta_{t4}^*+\beta_{t5}^*\right),\left(\beta_1^*+\beta_2^*+\beta_3^*+\beta_4^*+\beta_5^*+\beta_6^*\right),\left(\xi_{h2}+\psi_h+\mu_h\right),\left(\xi_{h1}+\psi_t+\mu_h\right)\\,\left(\psi_{th}+\mu_h\right),\left(\omega_{h2}+\rho_h+\delta_h+\mu_h\right),\ \left(\gamma_A+\delta_A+\mu_h\right),\ \left(\gamma_t+\delta_t+\mu_h\right),\left(\rho_{th}+\delta_{th}+\mu_h\right)\\\left(\delta_h+\mu_h\right),\left(\phi_t+\delta_t+\mu_h\right),\left(\alpha_{1th}+\alpha_{2th}+\delta_6+\mu_h\right),\left(\eta+\mu_h\right),\left(\mu_h\right)\end{pmatrix}.$$

…(19).

Lemma 2. Eq. shows the first and second terms for (6), (7) of the initial value problem. The $\mathrm{X}(t)\in A_c^0\left(f\right).$ mathematical equation (19) exists and will only have a single solution.

Using Picard-Lindelof and fixed point theory, we consider the solution of

$$\mathrm{X}\left(t\right)=S_h\left(\mathrm{X}\left(t\right)\right),$$

where S is expressed as the Picard operator:

$$S_h:A_c^0\left(f,R_+^{13}\right)\rightarrow A_c^0\left(f,R_+^{13}\right).$$

Therefore,

$$S_h\left(X\left(t\right)\right)=X\left(0\right)+\frac{1}{\Gamma\left(\gamma\right)}\int_0^t\left(t-\lambda\right)^{\gamma-1}Z\left(\lambda,\mathrm{X}\left(\lambda\right)\right)\mathrm{d}\lambda.$$

which becomes

$$\left\|S_h\left(\mathrm{X}_1\left(t\right)\right)-S_h\left(\mathrm{X}_2\left(t\right)\right)\right\|$$

$$=\left\|\frac{1}{\Gamma\left(\gamma\right)}\left[\int_0^t\left(t-\lambda\right)^{\gamma-1}Z\left(\lambda,\mathrm{X}_1\left(\lambda\right)\right)-Z\left(\lambda,\mathrm{X}_2\left(\lambda\right)\right)\mathrm{d}\lambda\right]\right\|$$

$$\leq\frac{1}{\Gamma\left(\gamma\right)}\int_0^t\left(t-\lambda\right)^{\gamma-1}\left\|Z\left(\lambda,\mathrm{X}_1\left(\lambda\right)\right)-Z\left(\lambda,\mathrm{X}_2\left(\lambda\right)\right)\mathrm{d}\lambda\right\|.$$

$$\leq\frac{\beta}{\Gamma\left(\gamma\right)}\int_0^t\left(t-\lambda\right)^{\gamma-1}\left\|\mathrm{X}_1-\mathrm{X}_2\right\|d\lambda.$$

$$\left\| S_h\left(\mathrm{X}_1(t)\right) - S_h\left(\mathrm{X}_2(t)\right) \right\| \le \frac{\beta}{\Gamma(\gamma+1) S_h}.$$

When $\frac{\beta}{\Gamma(\gamma+1)} S_h \le 1$,Picard's operator contradicting the conditions of existence, so any solution given by Eq.(6), (7) is unique.

### 5.9 Basic Reproduction number of the co-infection model

The Basic reproduction number of Typhoid fever and HIV/AIDs co-infection is the maximum of their the basic reproduction.

$$R_0^{ht} = \max\left(R_0^h, R_0^t\right)$$

$$R_0^{ht} = \max\left( \frac{\psi_h\left(A_3A_4\beta_1 + A_3\beta_3\gamma_h + A_4\beta_2\rho_h + \beta_3\gamma_A\rho_h\right)}{A_3A_2A_1A_4}, \frac{\beta_t\psi_t}{\left(\psi_t+\mu_h\right)\left(\gamma_t+\delta_t+\mu_h\right)} \right).$$

where $A_1 = \left(\psi_h + \mu_h\right), A_2 = \left(\rho_h + \gamma_h + \delta_h + \mu_h\right), A_3 = \left(\gamma_A + \delta_A + \mu_h\right), A_4 = \left(\delta_h + \mu_h\right).$

### 5.10 Global stability analysis at endemic equilibrium state

Equilibrium stability around the world is checked with the direct Lyapunov method. An endemic equilibrium point is considered globally stable when $R_0 < 1$. Epidemiology states that the disease will infect the whole population regardless of how many were exposed to it initially. from the fractional model (6).

Where $\lambda_t = \frac{\left(\beta_{t1}I_t + \beta_{t2}I_{th} + \beta_{t3}I_{tA} + \beta_{t4}B\right)}{N_h}$ shows the infection rate that results from the way typhoid is spread. $\lambda_h = \frac{\left(\beta_1 I_h + \beta_2 I_A + \beta_3 T_h + \beta_4 I_{tA} + \beta_5 T_{th} + \beta_6 I_{th}\right)}{N_h}$ shows the infection rate that results from the way HIV/AIDS is spread.

Where $N_h = \frac{\Lambda_h}{\mu_h}$ $as\, t \to \infty$,

our fractional model now becomes

$$^cD_t^\gamma S_h = \Lambda_h + \eta R_t - \lambda_h S_h - \lambda_t S_h - P_1 S_h,$$

$$^cD_t^\gamma E_t = \lambda_t S_h - \lambda_h \xi_{t1} E_t - P_2 E_t,$$

$$^cD_t^\gamma E_h = \lambda_h S_h - \lambda_t \xi_{t2} E_h - P_3 E_h,$$

$$^cD_t^\gamma E_{th} = \lambda_t \xi_{h2} E_h + \lambda_h \xi_{t1} E_t - P_4 E_{th},$$

$$^cD_t^\gamma I_h = \psi_{h1} E_h - P_5 I_h,$$

$$
\begin{aligned}
{}^{c}D_t^{\gamma} I_A &= \rho_h I_h - P_6 I_A,\\
{}^{c}D_t^{\gamma} I_t &= \psi_t E_t - P_7 I_t,\\
{}^{c}D_t^{\gamma} I_{th} &= \psi_{th} E_{th} + \psi_{h2} I_h + \psi_{t1} I_t - P_8 I_{th},\\
{}^{c}D_t^{\gamma} T_h &= \gamma_A I_A + \alpha_{1th} T_{th} - P_9 T_h,\\
{}^{c}D_t^{\gamma} T_t &= \gamma_t I_t + \alpha_{2th} T_{th} - P_{10} T_t,\\
{}^{c}D_t^{\gamma} T_{th} &= \rho_{th} I_{th} - P_{11} T_{th},\\
{}^{c}D_t^{\gamma} R_t &= \phi_t T_t - P_{12} R_t,\\
{}^{c}D_t^{\gamma} B &= \pi - P_{13} B.
\end{aligned}
\tag{20}
$$

When the physical system reaches the equilibrium point Eq. As a result, (24) produces the following outcomes:

$$\mu_h S_h^* - \eta R_t^* = \Lambda_h, \left(\psi_t + \mu_h\right) E_t^* = \lambda_t S_h^*, \left(\psi_h + \mu_h\right) E_h^* = \lambda_h S_h^*, \left(\psi_{th} + \mu_h\right) E_{th}^* = \lambda_h \xi_{t1} E_t^*,$$
$$\left(\omega_{h2} + \rho_h + \delta_h + \mu_h\right) I_h^* = \psi_{h1} E_h^*, \left(\gamma_A + \delta_A + \mu_h\right) I_A^* = \rho_h I_h^*, \left(\gamma_t + \delta_t + \mu_h\right) I_t^* = \psi_t E_t^*,$$
$$\left(\rho_{th} + \delta_{th} + \mu_h\right) I_{th}^* = \psi_{th} E_{th}^* + \psi_{h2} I_h^* + \psi_{t1} I_t^*, \left(\delta_h + \mu_h\right) T_h^* = \gamma_A I_A^* + \alpha_{1th} T_{th}^*,$$
$$\left(\phi_t + \delta_t + \mu_h\right) T_t^* = \gamma_t I_t^* + \alpha_{2th} T_{th}^*, \left(\alpha_{1th} + \alpha_{2th} + \delta_{th} + \mu_h\right) T_{th}^* = \rho_{th} I_{th}^*, \left(\eta + \mu_h\right) R_t^* = \phi_t T_t^*,$$
$$\mu_B B^* = \pi.$$

**Theorem 1.**

Model (24) is globally asymptotically stable if $R_0 < 1$ whenever

$$\left(13 - \frac{S_h^*}{S_h} + \frac{\Lambda_h}{\Lambda_h^*}\left(1 - \frac{S_h E_h^* E_t^* E_{th}^*}{S_h^* E_h E_t E_{th}}\right) - \frac{I_h^* I_A^* I_t^* I_{th}^* E_h^* E_t^* E_{th}^*}{I_h I_A I_t I_{th} E_h E_t E_{th}} - \frac{T_h T_t T_{th}}{T_h^* T_h^* I_{TC}^*} - \frac{T_{th}^* I_h I_A I_t I_{th}}{I_h^* I_A^* I_t^* I_{th}^*}\right) \leq 0.$$

*Let* $L(t) = L_{th}(t)$

be a non-linear Lyaponov function as stated in (27) below:

$$L(t)=L_1\left(S_h-S_h^*-S_h^*\ln\frac{S_h}{S_h^*}\right)+L_2\left(E_h-E_h^*-E_h^*\ln\frac{E_h}{E_h^*}\right)+L_3\left(E_t-E_t^*-E_t^*\ln\frac{E_t}{E_t^*}\right)$$
$$+L_4\left(E_{th}-E_{th}^*-E_{th}^*\ln\frac{E_{th}}{E_{th}^*}\right)+L_5\left(I_h-I_h^*-I_h^*\ln\frac{I_h}{I_h^*}\right)+L_6\left(I_A-I_A^*-I_A^*\ln\frac{I_A}{I_A^*}\right)$$
$$+L_7\left(I_t-I_t^*-I_t^*\ln\frac{I_t}{I_t^*}\right)+L_8\left(I_{th}-I_{th}^*-I_{th}\ln\frac{I_{th}}{I_{th}^*}\right)+L_9\left(T_h-T_h^*-T_h^*\ln\frac{T_h}{T_h^*}\right)$$
$$+L_{10}\left(T_t-T_t^*-T_t^*\ln\frac{T_t}{T_t^*}\right)+L_{11}\left(T_{th}-T_{th}^*-T_{th}^*\ln\frac{T_{th}}{T_{th}^*}\right)+L_{12}\left(R_t-R_t^*-R_t^*\ln\frac{R_t}{R_t^*}\right)$$
$$+L_{13}\left(B-B^*-B^*\ln\frac{B}{B^*}\right) \tag{21}$$

The fractional order derivative of Eq. (26), using Caputo, gives:

$${}^cD_t^\gamma L(t)={}^cD_t^\gamma L_{th}(t)\le L_1\left(1-\frac{S_h^*}{S_h}\right){}^cD_t^\gamma S_h(t)+L_2\left(1-\frac{E_h^*}{E_h}\right){}^cD_t^\gamma E_h(t)$$
$$+L_3\left(1-\frac{E_t^*}{E_t}\right){}^cD_t^\gamma E_t(t)+L_4\left(1-\frac{E_{th}^*}{E_{th}}\right){}^cD_t^\gamma E_{th}(t)+L_5\left(1-\frac{I_h^*}{I_h}\right){}^cD_t^\gamma I_h(t)$$
$$+L_6\left(1-\frac{I_A^*}{I_A}\right){}^cD_t^\gamma I_A(t)+L_7\left(1-\frac{I_t^*}{I_t}\right){}^cD_t^\gamma I_t(t)+L_8\left(1-\frac{I_{th}^*}{I_{th}}\right){}^cD_t^\gamma I_{th}(t)$$
$$+L_9\left(1-\frac{T_h^*}{T_h}\right){}^cD_t^\gamma T_h(t)+L_{10}\left(1-\frac{T_t^*}{T_t}\right){}^cD_t^\gamma T_t(t)+L_{11}\left(1-\frac{T_{th}^*}{T_{th}}\right){}^cD_t^\gamma T_{th}(t)$$
$$+L_{12}\left(1-\frac{R_t^*}{R_t}\right){}^cD_t^\gamma R_t(t)+L_{13}\left(1-\frac{B^*}{B}\right){}^cD_t^\gamma B(t).$$

$${}^cD_t^\gamma L(t)=\frac{1}{\Lambda^* S_h^*}\left(\begin{array}{l}\left(1-\frac{S_h^*}{S_h}\right){}^cD_t^\gamma S_h(t)+\left(1-\frac{E_h^*}{E_h}\right){}^cD_t^\gamma E_h(t)+\left(1-\frac{E_t^*}{E_t}\right){}^cD_t^\gamma E_t(t)\\+\left(1-\frac{E_{th}^*}{E_{th}}\right){}^cD_t^\gamma E_{th}(t)+\left(1-\frac{I_h^*}{I_h}\right){}^cD_t^\gamma I_h(t)+\left(1-\frac{I_A^*}{I_A}\right){}^cD_t^\gamma I_A(t)\\+\left(1-\frac{I_t^*}{I_t}\right){}^cD_t^\gamma I_t(t)+\left(1-\frac{I_{th}^*}{I_{th}}\right){}^cD_t^\gamma I_{th}(t)+\left(1-\frac{T_h^*}{T_h}\right){}^cD_t^\gamma T_h(t)\\+\left(1-\frac{T_t^*}{T_t}\right){}^cD_t^\gamma T_t(t)+\left(1-\frac{T_{th}^*}{T_{th}}\right){}^cD_t^\gamma T_{th}(t)+\left(1-\frac{R_t^*}{R_t}\right){}^cD_t^\gamma R_t(t)\\+\left(1-\frac{B^*}{B}\right){}^cD_t^\gamma B(t)\end{array}\right).$$

$$\left(1-\frac{S_h^*}{S_h}\right){}^cD_t^\gamma S_h=\left(1-\frac{S_h^*}{S}\right)\left(\Lambda_{h1}^* S_h^*+\mu S_h^*-\lambda_1 S_h-\mu S_h\right),$$

$$= \Lambda_{h1}^{*} S_h^{*} \left(1 - \frac{S_h \Lambda_{h1}}{\lambda_1^{*} S_h^{*}} - \frac{S_h^{*}}{S} + \frac{\Lambda_{h1}}{\Lambda_{h1}^{*}}\right) + \mu_h S_h^{*} \left(2 - \frac{S_h}{S_h^{*}} - \frac{S_h^{*}}{S_h}\right),$$

$$\left(1 - \frac{E_h^{*}}{E_h}\right) {}^{c}D_t^{\gamma} E_h = \left(1 - \frac{E_h^{*}}{E_h}\right)\left(\Lambda_{h1}^{*} S_h - \Lambda_{h1}^{*} S_h^{*} \frac{E_h}{E_h^{*}}\right),$$

$$= \Lambda_{h1}^{*} S_h \left(1 - \frac{S_h \Lambda_{h1}^{*}}{\Lambda_{h1}^{*} S_h^{*}} \frac{E_h^{*}}{E_h} - \frac{E_h^{*}}{E_h} + \frac{S_h^{*} \Lambda_1}{\Lambda_{h1}^{*} S_h^{*}}\right),$$

$$\left(1 - \frac{E_t^{*}}{E_t}\right) {}^{c}D_t^{\gamma} E_t = \left(1 - \frac{E_t^{*}}{E_t}\right)\left(\Lambda_{h1}^{*} S_h - \Lambda_{h1}^{*} S_h^{*} \frac{E_t}{E_t^{*}}\right),$$

$$= \Lambda_{h1}^{*} S_h^{*} \left(1 - \frac{S_h \lambda_1^{*}}{\Lambda_{h1}^{*} S_h^{*}} \frac{E_t^{*}}{E_t} - \frac{E_t^{*}}{E_t} + \frac{S_h^{*} \Lambda_{h1}}{\Lambda_{h1}^{*} S_h^{*}}\right),$$

$$\left(1 - \frac{E_{th}^{*}}{E_{th}}\right) {}^{c}D_t^{\gamma} E_{th} = \left(1 - \frac{E_{th}^{*}}{E_{th}}\right)\left(\Lambda_{h1}^{*} S - \Lambda_{h1}^{*} S_h^{*} \frac{E_{th}}{E_{th}^{*}}\right),$$

$$= \Lambda_{h1}^{*} S_h^{*} \left(1 - \frac{S_h \Lambda_{h1}^{*}}{\Lambda_{h1}^{*} S_h^{*}} \frac{E_{th}^{*}}{E_{th}} - \frac{E_{th}^{*}}{E_{th}} + \frac{S_h^{*} \Lambda_{h1}}{\Lambda_{h1}^{*} S_h^{*}}\right),$$

$$\frac{P_1}{\rho_h}\left(1 - \frac{I_h^{*}}{I_h}\right) {}^{c}D_t^{\gamma} I_h = \frac{P_1}{\rho_h}\left(1 - \frac{I_h^{*}}{I_h}\right)\left(\psi_h E_h - P_2 \frac{I_h}{I_h^{*}} I_h^{*}\right),$$

$$= \Lambda_1^{*} S_h^{*} \left(1 - \frac{E_h E_t E_{th}}{E_h^{*} E_{th}^{*} E_t^{*}} - \frac{I_h I_t}{I_h^{*} I_t^{*}} - \frac{E_h E_t E_{th} I_h^{*}}{E_h^{*} I_h}\right),$$

$$\frac{P_2}{\psi_h}\left(1 - \frac{I_A^{*}}{I_A}\right) {}^{c}D_t^{\gamma} I_A = \frac{P_2}{\psi_h}\left(1 - \frac{I_A^{*}}{I_A}\right)\left(\gamma_A E_H - P_3 \frac{I_A}{I_A^{*}} I_A^{*}\right), \tag{22}$$

$$= \Lambda_{h1}^{*} S^{*} \left(1 - \frac{E_h E_t E_{th}}{E_h^{*} E_{th}^{*} E_t^{*}} - \frac{I_t}{I_t^{*}} - \frac{E_h E_t E_{th} I_t^{*}}{E_h^{*} I_t}\right),$$

$$\frac{P_3}{\psi_t}\left(1 - \frac{I_t^{*}}{I_t}\right) {}^{c}D_t^{\gamma} I_t = \frac{P_3}{\psi_t}\left(1 - \frac{I_t^{*}}{I_t}\right)\left(\omega_2 E_t - P_4 \frac{I_t}{I_t^{*}} I_t^{*}\right),$$

$$= \Lambda_{h1}^{*} S_h^{*} \left(1 - \frac{E_h E_t E_{th}}{E_h^{*} E_{th}^{*} E_t^{*}} - \frac{I_t}{I_t^{*}} - \frac{E_h E_t E_{th} I_t^{*}}{E_h^{*} I_t}\right),$$

$$\frac{P_4}{\rho_h}\left(1 - \frac{I_{th}^{*}}{I_{th}}\right) {}^{c}D_t^{\gamma} I_{th} = \frac{P_4}{\rho_h}\left(1 - \frac{I_{th}^{*}}{I_{th}}\right)\left(\psi_{th} E_{th} - P_5 \frac{I_{th}}{I_{th}^{*}} I_{th}^{*}\right),$$

$$= \Lambda_{h1}^{*} S_h^{*} \left( 1 - \frac{E_h E_t E_{th}}{E_h^{*} E_{th}^{*} E_t^{*}} - \frac{I_{th}}{I_{th}^{*}} - \frac{E_h E_t E_{th} I_{th}^{*}}{E_h^{*} I_{th}} \right),$$

$$\frac{P_8 P_9}{\alpha_{1th}\alpha_{2th}} \left( 1 - \frac{T_h^{*}}{T_h} \right) {}^{c}D_t^{\gamma} T_h = \frac{P_8 P_9}{\alpha_{1th}\alpha_{2th}} \left( 1 - \frac{T_h^{*}}{T_h} \right) \left( \alpha_{1th}\alpha_{2th} I_h I_A I_t - P_9 \frac{T_h}{T_h^{*}} T_h^{*} \right),$$

$$= \Lambda_{h1}^{*} S_h^{*} \left( 1 - \frac{T_h^{*}}{T_h} - \frac{I_h I_A I_t I_{th} T_h^{*}}{T_h^{*} I_{th}^{*} I_h^{*} I_{th}^{*} I_t^{*}} + \frac{I_h I_A I_t I_{th}}{I_{th}^{*} I_h^{*} I_{th}^{*} I_t^{*}} \right),$$

Hence, Eq. (25) now becomes;

$$\left( 13 - \frac{S_h^{*}}{S_h} + \frac{\Lambda_h}{\Lambda_h^{*}} \left( 1 - \frac{S_h E_h^{*} E_t^{*} E_{th}^{*}}{S_h^{*} E_h E_t E_{th}} \right) - \frac{I_h^{*} I_A^{*} I_t^{*} I_{th}^{*} E_h^{*} E_t^{*} E_{th}^{*}}{I_h I_A I_t I_{th} E_h E_t E_{th}} - \frac{T_h T_t T_{th}}{T_h^{*} T_h^{*} I_{TC}^{*}} - \frac{T_{th}^{*} I_h I_A I_t I_{th}}{I_h^{*} I_A^{*} I_t^{*} I_{th}^{*}} \right) \leq 0.$$

$${}^{c}D_t^{\gamma} L(t) \leq \Lambda_{h1}^{*} S_h^{*}$$

Which implies that,

$${}^{c}D_t^{\gamma} L(t) \leq \Lambda_{h1}^{*} \rho_h (1 - R_0) \Lambda_h S_h^{*}$$

$$-\rho_h (1 - R_0) \Lambda_h S_h^{*} \left[ P_2 S_h^{*} \left( \frac{S_h^{*}}{S_h} - 1 - \ln \frac{S_h^{*}}{S_h} \right) \right]$$

Therefore ${}^{c}D_t^{\gamma} L(t) \leq 0 \quad for \ R_0 < 1.$

This implies that, ${}^{c}D_t^{\gamma} L(t) = 0$.

Since $E_{*} = \left( S^{*}, E_h^{*}, E_t^{*}, E_{th}^{*}, I_h^{*}, I_A^{*}, I_t^{*}, I_{th}^{*}, T_h^{*}, T_t^{*}, T_{th}^{*}, \mathrm{R}_t^{*}, \mathrm{B}^{*} \right)$ is the endemic equilibrium point, by LaSalle's invariance principle, is globally asymptotically stable in $\Omega$ whenever $R_0 < 1.$

## 5.11 Fractional order model numerical results

With the generalized fractional Adams-Bashforth-Moulton scheme mentioned in [25], we executed a numerical study of the HIV/AIDs and Typhoid fever co-infection model. The value of the parameters used in the model are shown in Table 1 which also displays simulated results of different fractional order $(\gamma)$.

## 5.12 Implementation of fractional Adams–Bashforth–Moulton method

This study has followed the process outlined in [25,26]. Fomr (6) we solved the HIV/AIDs and Typhoid fever fractional co-infection problem using the fractional Adams-Bashforth-Moulton method. Now, the fractional model is represented by:

$${}^{c}D_t^{\gamma} P(t) = Q(t, \mathrm{q}(t)), \ 0 < t < \beta, \quad (23)$$

$$P^{(n)}(0) = P_0^{(n)}, \ n = 1, 0, \ldots, \mathrm{q}, \mathrm{q} = [\gamma].$$

Where $P=\left(S^*,E_h^*,E_t^*,E_{th}^*,I_h^*,\ I_A^*,I_t^*,I_{th}^*,T_h^*,T_t^*,T_{th}^*,\mathrm{R}_t^*,\mathrm{B}^*\right)\in R_+^{13}$ and $V\left(t,\mathrm{q}\left(t\right)\right)$ means the function is real and it is continuous.

Thus, equation (23) can be expressed with fractional integral as below:

$$P\left(t\right)=\sum_{n=0}^{m-1}P_0^{(n)}\frac{t^n}{n!}+\frac{1}{\Gamma\left(\gamma\right)}\int_0^t\left(t-y\right)^{\gamma-1}R\left(\mathrm{k},m\left(k\right)\right)dk \tag{24}$$

Using the technique defined in [25], we set the step size $g=\frac{\beta}{N}$, $N\in\mathrm{N}$ with a grid that is uniform on $\left[0,\beta\right]$. Where $t_c=cr,\ c=0,1,1,...N.$ For this reason, the fractional order model for HIV/AIDs and Typhoid fever co-infection mentioned in (6) can be interpreted as:

$$\begin{aligned}
&S_{hk+1}\left(t\right)=S_{h0}+\frac{g^\gamma}{\Gamma\left(\gamma+2\right)}\left\{\Lambda_h+\eta R_{t\ t}^n-\lambda_hS_h^n-\lambda_tS_h^n-P_1S_h^n,\right\}+\\
&\frac{g^\gamma}{\Gamma\left(\gamma+2\right)}\sum_{y=0}^{k}dy,k+1\left\{\Lambda_h+\eta R_{ty}-\lambda_hS_{hy}-\lambda_tS_{hy}-P_1S_{hy}\right\}\\
&E_{hk+1}\left(t\right)=E_{h0}+\frac{g^\gamma}{\Gamma\left(\gamma+2\right)}\left\{\lambda_tS_h^n-\lambda_h\xi_{t1}E_t^n-P_2E_t^n\right\}+\\
&\frac{g^\gamma}{\Gamma\left(\gamma+2\right)}\sum_{y=0}^{k}dy,k+1\left\{\lambda_tS_{hy}-\lambda_h\xi_{t1}E_{ty}-P_2E_{ty}\right\},\\
&E_{tk+1}\left(t\right)=E_{t0}+\frac{g^\gamma}{\Gamma\left(\gamma+2\right)}\left\{\lambda_hS_h^n-\lambda_t\xi_{t2}E_h^n-P_3E_h^n\right\}+\\
&\frac{g^\gamma}{\Gamma\left(\gamma+2\right)}\sum_{y=0}^{k}dy,k+1\left\{\lambda_hS_{hy}-\lambda_t\xi_{t2}E_{hy}-P_3E_{hy}\right\},\\
&E_{thk+1}\left(t\right)=E_{th0}+\frac{g^\gamma}{\Gamma\left(\gamma+2\right)}\left\{\lambda_t\xi_{h2}E_h^n+\lambda_h\xi_{t1}E_t^n-P_4E_{th}^n\right\}+\\
&\frac{g^\gamma}{\Gamma\left(\gamma+2\right)}\sum_{y=0}^{k}dy,k+1\left\{\lambda_t\xi_{h2}E_{hy}+\lambda_h\xi_{t1}E_{ty}-P_4E_{thy}\right\},\\
&I_{hk+1}\left(t\right)=I_{h0}+\frac{g^\gamma}{\Gamma\left(\gamma+2\right)}\left\{\psi_{h1}E_h^n-P_5I_h^n\right\}+\\
&\frac{g^\gamma}{\Gamma\left(\gamma+2\right)}\sum_{y=0}^{k}dy,k+1\left\{\psi_{h1}E_{hy}-P_5I_{hy}\right\},
\end{aligned} \tag{25}$$

$$I_{Ak+1}(t) = I_{A0} + \frac{g^{\gamma}}{\Gamma(\gamma+2)}\left\{\rho_h I_h^n - P_6 I_A^n\right\} +$$

$$\frac{g^{\gamma}}{\Gamma(\gamma+2)}\sum_{y=0}^{k} dy, k+1\left\{\rho_h I_{hy} - P_6 I_{Ay}\right\},$$

$$I_{tk+1}(t) = I_{t0} + \frac{g^{\gamma}}{\Gamma(\gamma+2)}\left\{\psi_t E_t^n - P_7 I_t^n\right\} +$$

$$\frac{g^{\gamma}}{\Gamma(\gamma+2)}\sum_{y=0}^{k} dy, k+1\left\{\psi_t E_{ty} - P_7 I_{ty}\right\},$$

$$I_{thk+1}(t) = I_{th0} + \frac{g^{\gamma}}{\Gamma(\gamma+2)}\left\{\psi_{th} E_{th}^n + \psi_{h2} I_h^n + \psi_{t1} I_t^n - P_8 I_{th}^n\right\} +$$

$$\frac{g^{\gamma}}{\Gamma(\gamma+2)}\sum_{y=0}^{k} dy, k+1\left\{\psi_{th} E_{thy} + \psi_{h2} I_{hy} + \psi_{t1} I_{ty} - P_8 I_{thy}\right\},$$

$$T_{hk+1}(t) = T_{h0} + \frac{g^{\gamma}}{\Gamma(\gamma+2)}\left\{\gamma_A I_A^n + \alpha_{1th} T_{th}^n - P_9 T_h^n\right\} +$$

$$\frac{g^{\gamma}}{\Gamma(\gamma+2)}\sum_{y=0}^{k} dy, k+1\left\{\gamma_A I_{Ay} + \alpha_{1th} T_{thy} - P_9 T_{hy}\right\},$$

$$T_{tk+1}(t) = T_{t0} + \frac{g^{\gamma}}{\Gamma(\gamma+2)}\left\{\gamma_t I_t^n + \alpha_{2th} T_{th}^n - P_{10} T_t^n\right\} +$$

$$\frac{g^{\gamma}}{\Gamma(\gamma+2)}\sum_{y=0}^{k} dy, k+1\left\{\gamma_t I_{ty} + \alpha_{2th} T_{thy} - P_{10} T_{ty}\right\},$$

$$T_{thk+1}(t) = T_{th0} + \frac{g^{\gamma}}{\Gamma(\gamma+2)}\left\{\rho_{th} I_{th}^n - P_{11} T_{th}^n\right\} +$$

$$\frac{g^{\gamma}}{\Gamma(\gamma+2)}\sum_{y=0}^{k} dy, k+1\left\{\rho_{th} I_{thy} - P_{11} T_{thy}\right\},$$

$$R_{tk+1}(t) = R_{t0} + \frac{g^{\gamma}}{\Gamma(\gamma+2)}\left\{\phi_t T_t^n - P_{12} R_t^n\right\} +$$

$$\frac{g^{\gamma}}{\Gamma(\gamma+2)}\sum_{y=0}^{k} dy, k+1\left\{\phi_t T_{ty} - P_{12} R_{ty}\right\},$$

$$B_{k+1}(t) = B_0 + \frac{g^{\gamma}}{\Gamma(\gamma+2)}\left\{\pi - P_{13} B^n\right\} +$$

$$\frac{g^{\gamma}}{\Gamma(\gamma+2)}\sum_{y=0}^{k} dy, k+1\left\{\pi - P_{13} B_y\right\},$$

Where

$$S_{hk+1}^{n}(t)=S_{h0}+\frac{1}{\Gamma(\gamma)}\sum_{y=0}^{k}f_{y,k+1}\left\{\Lambda_h+\eta R_{ty}-\lambda_h S_{hy}-\lambda_t S_{hy}-P_1 S_{hy}\right\},$$

$$E_{hk+1}^{n}(t)=E_{h0}+\frac{1}{\Gamma(\gamma)}\sum_{y=0}^{k}f_{y,k+1}\left\{\lambda_t S_{hy}-\lambda_h \xi_{t1} E_{ty}-P_2 E_{ty}\right\},$$

$$E_{tk+1}^{n}(t)=E_{t0}+\frac{1}{\Gamma(\gamma)}\sum_{y=0}^{k}f_{y,k+1}\left\{\lambda_h S_{hy}-\lambda_t \xi_{t2} E_{hy}-P_3 E_{hy}\right\}, \quad (26)$$

$$E_{thk+1}^{n}(t)=E_{HC0}+\frac{1}{\Gamma(\gamma)}\sum_{y=0}^{k}f_{y,k+1}\left\{\lambda_t \xi_{h2} E_{hy}+\lambda_h \xi_{t1} E_{ty}-P_4 E_{thy}\right\},$$

$$I_{hk+1}^{n}(t)=I_{h0}+\frac{1}{\Gamma(\gamma)}\sum_{y=0}^{k}f_{y,k+1}\left\{\psi_{h1} E_{hy}-P_5 I_{hy}\right\},$$

$$I_{Ak+1}^{n}(t)=I_{A0}+\frac{1}{\Gamma(\gamma)}\sum_{y=0}^{k}f_{y,k+1}\left\{\rho_h I_{hy}-P_6 I_{Ay}\right\},$$

$$I_{tk+1}^{n}(t)=I_{t0}+\frac{1}{\Gamma(\gamma)}\sum_{y=0}^{k}f_{y,k+1}\left\{\psi_t E_{ty}-P_7 I_{ty}\right\},$$

$$I_{thk+1}^{n}(t)=I_{th0}+\frac{1}{\Gamma(\gamma)}\sum_{y=0}^{k}f_{y,k+1}\left\{\psi_{th} E_{thy}+\psi_{h2} I_{hy}+\psi_{t1} I_{ty}-P_8 I_{thy}\right\},$$

$$T_{hk+1}^{n}(t)=T_{h0}+\frac{1}{\Gamma(\gamma)}\sum_{y=0}^{k}f_{y,k+1}\left\{\gamma_A I_{Ay}+\alpha_{1th} T_{thy}-P_9 T_{hy}\right\},$$

$$T_{tk+1}^{n}(t)=T_{t0}+\frac{1}{\Gamma(\gamma)}\sum_{y=0}^{k}f_{y,k+1}\left\{\gamma_t I_{ty}+\alpha_{2th} T_{thy}-P_{10} T_{ty}\right\},$$

$$T_{thk+1}^{n}(t)=T_{th0}+\frac{1}{\Gamma(\gamma)}\sum_{y=0}^{k}f_{y,k+1}\left\{\rho_{th} I_{thy}-P_{11} T_{thy}\right\},$$

$$R_{Ck+1}^{n}(t)=R_{C0}+\frac{1}{\Gamma(\gamma)}\sum_{y=0}^{k}f_{y,k+1}\left\{\phi_t T_{ty}-P_{12} R_{ty}\right\},$$

$$B_{k+1}^{n}(t)=B_0+\frac{1}{\Gamma(\gamma)}\sum_{y=0}^{k}f_{y,k+1}\left\{\pi-P_{13} B_y\right\}.$$

From (29) and (30) obtained;

$$dy,_{K+1}=K^{\gamma+1}-(k-\gamma)(k+\gamma)^{\gamma},\ y=0$$

$$(k-y+2)^{\gamma+1}+(k-\gamma)^{\gamma+1}-2(k-y+1)^{\gamma+1},\ 1\le y\le k$$

and $f_{y,k+1} = \frac{g^{\gamma}}{\gamma}\left[(k-y+1)^{\gamma}(k-y)^{\gamma}\right]$, $0 \le y \le k$.

## 5.13 Importance of using the fractional Adam-Bashforth Moulton method in obtaining the numerical solutions of the model

1. This method needs one more evaluation of the function at each point, but it still assures high-order accuracy.

2. This way, errors are handled automatically and many ODE solvers use this approach.

3. It is widely used in engineering, chemistry and medicine because it is able to solve numerical partial and fractional-order differential equations.

## 5.14 Numerical Simulation

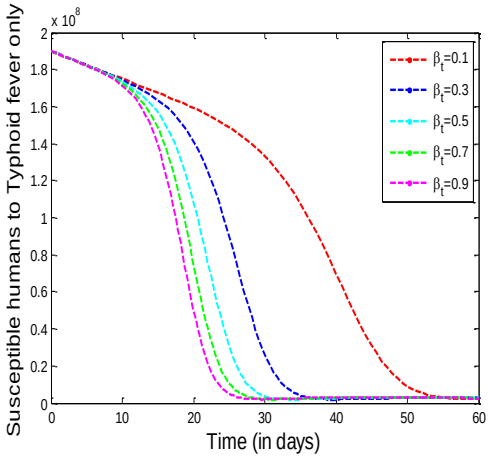


**Fig.3a: Susceptible humans to typhoid fever only**

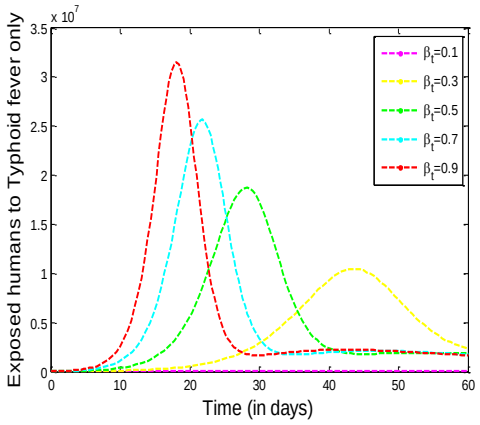


**Fig.3b: Exposed humans to typhoid fever only**

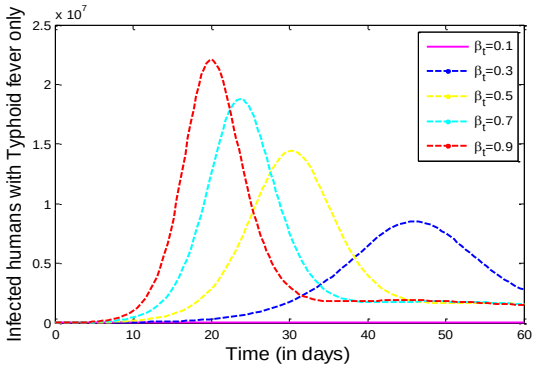


**Fig.3c: Infected humans with typhoid fever only**

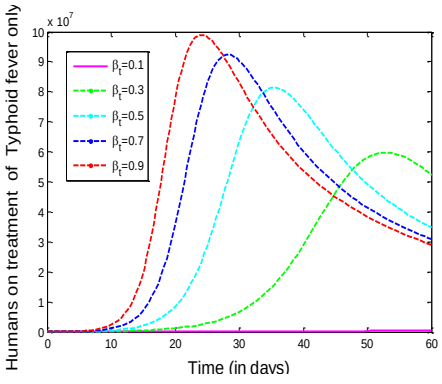


**Fig.3d: Humans on treatment of typhoid fever only**

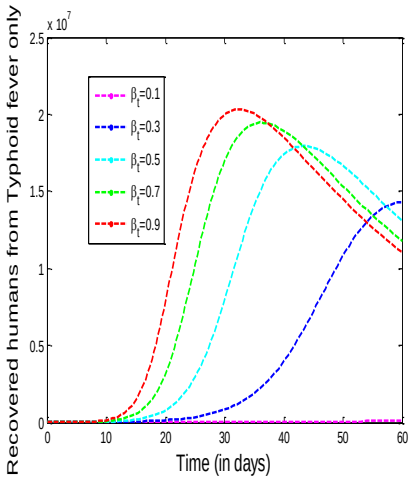


**Fig.3e: Recovered humans from typhoid fever onlv**

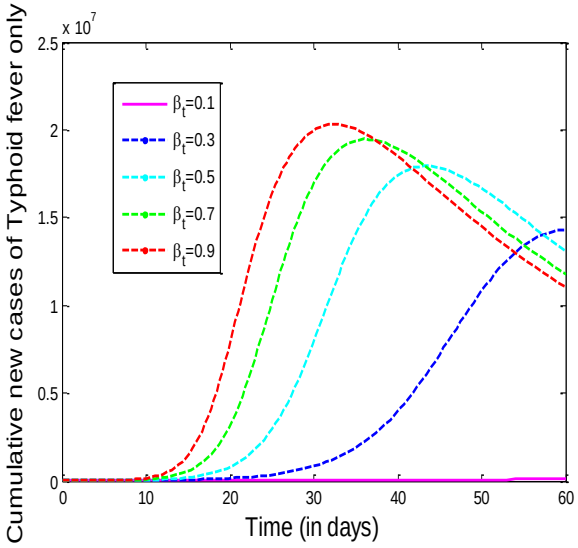


**Fig.3f: Cumulative new cases of typhoid fever only**

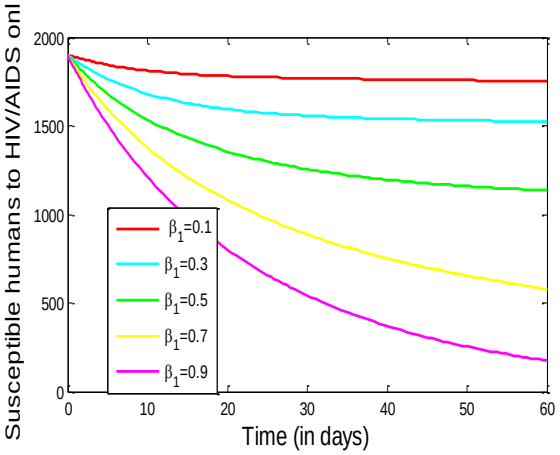


**Fig.4a: Susceptible humans to HIV/AIDS only**

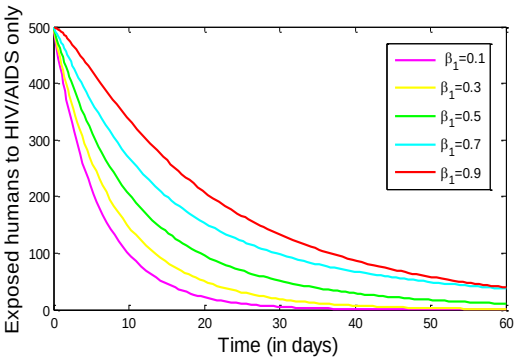


**Fig.4b: Exposed humans to HIV/AIDS only**

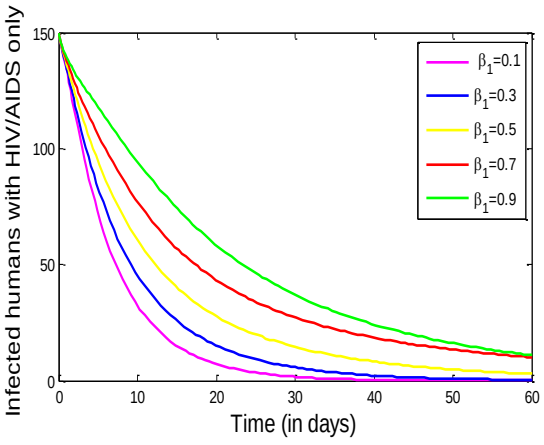


**Fig.4c: Infected humans with HIV/AIDS only**

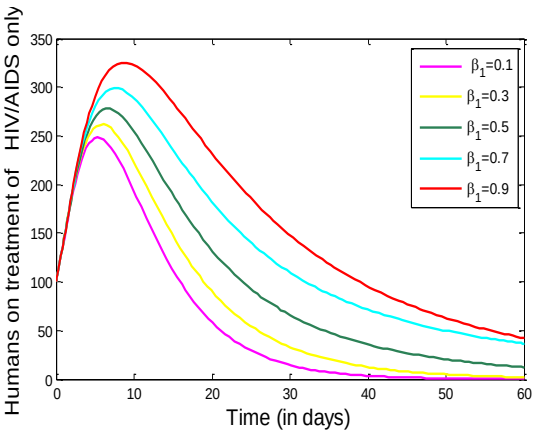

**Fig.4d: Humans on treatment of HIV/AIDS only**

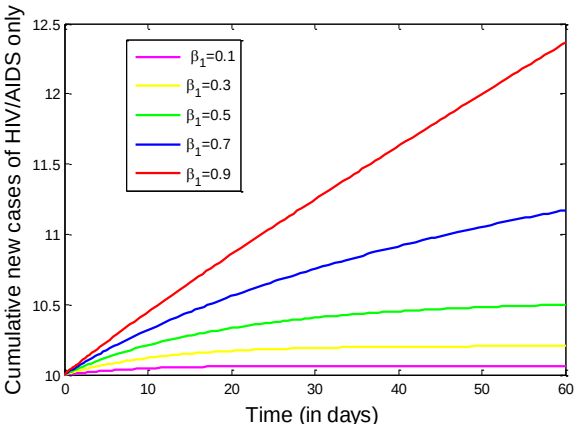


**Fig.4a: Cumulative new cases of HIV/AIDS only**

(3a) shows what happens in a susceptible population based on contact rate $(\beta_t)$, It is observed that as contact rate $(\beta_t)$ increases lead to the decrease in the number of people who are susceptible to the typhoid fever only. (3b) shows what happens in a Exposed human population based on contact rate $(\beta_t)$, It is observed that as contact rate $(\beta_t)$ increases lead to the increase in the number of people who are Exposed to the typhoid fever only. (3c) shows what happens in a infected human population based on contact rate $(\beta_t)$, It is observed that as contact rate $(\beta_t)$ increases lead to the increase in the number of people who are infected with typhoid fever only.

(3d) shows what happens in the population of human on Typhoid fever only treatment based on contact rate $(\beta_t)$, It is observed that as contact rate $(\beta_t)$ increases lead to the increase in the population of human on Typhoid fever only treatment. (3e) shows what happens in the population of Recovered humans from Typhoid fever only based on contact rate $(\beta_t)$, It is observed that as contact rate $(\beta_t)$ increases lead to the increase in the Recovered human population from typhoid fever only.(3f) shows what happens in the cumulative new cases of Typhoid fever only based on contact rate $(\beta_t)$, it is observed that as contact rate $(\beta_t)$ increases lead to the increase in the cumulative new cases of Typhoid fever only.(4a) shows what happens in a susceptible population based on contact rate $(\beta_1)$, It is observed that as contact rate $(\beta_1)$ increases lead to the decrease in the number of people who are susceptible to the HIV/AIDS only. (4b) shows what happens in a Exposed human population based on contact rate $(\beta_1)$, It is observed that as contact rate $(\beta_1)$ increases lead to the increase in the number of people who are Exposed to the HIV/AIDS only. (4c) shows what happens in a infected human population based on contact rate $(\beta_1)$, It is observed that as contact rate $(\beta_1)$ increases lead to the increase in the number of people who are infected with HIV/AIDS only.(4d) shows what happens in the population of human on HIV/AIDS only treatment

based on contact rate $(\beta_1)$, It is observed that as contact rate $(\beta_1)$ increases lead to the increase in the population of human on HIV/AIDS only treatment. (4e) shows what happens in the population of Recovered humans from HIV/AIDS only based on contact rate $(\beta_1)$, It is observed that as contact rate $(\beta_1)$ increases lead to the increase in the Recovered human population from HIV/AIDS only.(4f) shows what happens in the cumulative new cases of HIV/AIDS only based on contact rate $(\beta_1)$, it is observed that as contact rate $(\beta_1)$ increases lead to the increase in the cumulative new cases of HIV/AIDS only.

## 6.0 Conclusions

In this paper, a mathematical model is made using the Caputo fractional derivative to examine both the spread of Typhoid fever and HIV/AIDS co-infection, as well as the ways to control them. We conducted a wide-ranging theoretical analysis of this Typhoid fever and HIV/AIDS co-infection model because using fractional modeling is important for controlling such diseases. The model was resolved with the fractional Adams–Bashforth–Moulton numerical method. Model parameters and the used Caputo operator fractional orders determined how the simulations saw the diseases incidence. We studied contact rates between people who had Typhoid fever and HIV/AIDS co-infection. The findings indicated that cutting how often people are exposed to each other would help lower the number of Typhoid and HIV/AIDS co-infections among humans. According to [27], the research group must look for analytical solutions to non-linear partial differential equations.

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## Credit authorship contribution statement

Enejoh Jalija: Writing–original draft, Formal analysis. Jeremiah Amos: Writing – original draft, Formal analysis, William Atokolo Writing – original draft, Formal analysis., David Omale: Writing – original draft, Formal analysis., Emmanuel Abah: Writing – original draft., Ugbede Alih: Writing – original draft., Bolarinwa Bolaji: Writing – original draft, Formal analysis.

## Declaration of competing interest

No financial conflicts or personal relationships affecting the research findings of this paper exist according to the authors.

## Data availability

Values of parameters were fitted, and also parameters used are adequately cited and referenced.

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