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ON THE NORM OF ELEMENTARY OPERATOR OF LENGTH TWO IN TENSOR PRODUCT OF C*-ALGEBRAS

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Abstract:

Considerable research has been done on Norm property of different examples of Elementary operators with significant findings. From available literature not much have been done in determining the norm of elementary operator in tensor product of C*-algebras. The norms of Basic elementary operator, Jordan elementary operator and finite length elementary operator in tensor product of C*-algebras have been determined and results obtained. The main focus of this work is to investigate the norm of the elementary operator of length two in the tensor product of C*-algebras and to expand on our previous discussion on the elementary operator in tensor product of C*-algebras. To reach the goals, methods such as finite rank operator, tensor products of C*-algebras, and other well-known results were applied.

Keywords:

Elementary Operator of Length Two, Finite Rank Operator, Tensor Product of C*-algebras.

1. Introduction

Let $H \otimes K$ be tensor product of complex Hilbert spaces H and K and $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$. Let $A \otimes B, C \otimes D$ be fixed elements of $B(H \otimes K)$, with $A, C \in B(H)$ and $B, D \in B(K)$ are the sets of bounded linear operators on H and K, respectively. Muiruri et al. (2018) then provide us the following definitions:-

An elementary operator, $T_n: B(H \otimes K) \rightarrow B(H \otimes K)$, is defined as;

$$T_n(X \otimes Y) = \sum_{i=1}^{n} A_i \otimes B_i(X \otimes Y)C_i \otimes D_i, \forall X \otimes Y \in B(H \otimes K)$$

When n = 1 we obtain the basic elementary operator, $M_{A\otimes B, C\otimes D}: B(H \otimes K) \rightarrow B(H \otimes K)$, defined as;

$$M_{A\otimes B, C\otimes D}(X \otimes Y) = A \otimes B(X \otimes Y)C \otimes D, \forall X \otimes Y \in B(H \otimes K).$$

When n = 2 then we obtain an elementary operator of length two which is defined by;

$$T_2(X\otimes Y) = A_1 \otimes B_1(X \otimes Y)C_1 \otimes D_1 + A_2 \otimes B_2(X \otimes Y)C_2 \otimes D_2, \ \forall X \otimes Y \in B(H \otimes K).$$

The Jordan elementary operator, $U_{A\otimes B, C\otimes D}: B(H \otimes K) \rightarrow B(H \otimes K)$, is defined as;

$U_{A\otimes B, C\otimes D}(X \otimes Y) = A \otimes B(X \otimes Y)C \otimes D + C \otimes D(X \otimes Y)A \otimes B, \forall X \otimes Y \in B(H \otimes K).$

Aspart of our result, the notion of an elementary operator in the tensor product of C*-algebras can be expanded to encompass additional instances of elementary operators, as follows:

The left multiplication operator, $L_{A\otimes B}: B(H \otimes K) \rightarrow B(H \otimes K)$, is defined by

$$L_{A\otimes B}(X \otimes Y) = A \otimes B\,(X \otimes Y) \ \ \forall \ (X \otimes Y) \in B(H \otimes K).$$

The right multiplication operator, $R_{A\otimes B}: B(H \otimes K) \rightarrow B(H \otimes K)$, is defined by

$$R_{A\otimes B}(X \otimes Y) = (X \otimes Y)A \otimes B \ \ \forall \ (X \otimes Y) \in B(H \otimes K).$$

The general derivation operator is defined by

$$\delta_{A\otimes B, C\otimes D}(X\otimes Y) = A \otimes B(X \otimes Y) - (X \otimes Y)C \otimes D = L_{A\otimes B}(X \otimes Y) - R_{C\otimes D}(X \otimes Y).$$

The inner derivation operator is defined by

$$\delta_{A\otimes B,\ A\otimes B}(X\otimes Y) = A \otimes B(X \otimes Y) - (X \otimes Y)A \otimes B = L_{A\otimes B}(X \otimes Y) - R_{A\otimes B}(X \otimes Y).$$

2. Norm of Elementary Operator

King'ang'iet al. (2014) determined the norm of elementary operator of length two for finite-dimensional separable Hilbert space $W \in B(H)$ with $\| W \| = 1$ and $W(x) = x$ for all unit vectors $x \in H$ and proved theorem 2.1;

Theorem 2.1: (King'ang'iet al., 2014).

Let H be a complex Hilbert space and $B(H)$ be algebra of all bounded linear operators on H. Let E_2 be the elementary operator on $B(H)$. If for an operator $W \in B(H)$ with $\| W \| = 1$, we have $W(x) = x$ with all unit vectors $x \in H$, then $\| E_2 \| = \sum_{i=1}^{2} \| A_i \| \| B_i \|$.

King'ang'i (2017) employed the concept of the maximal numerical range of A^*B relative to B to determine the lower bound of the norm of an elementary operatorof length two and obtained theorem 2.2 ;

Theorem 2.2: (King'ang'i, 2017).

Let E_2 be an elementary operator of length two on $B(H)$. Then

$$\| E_2 \| \geq Sup_{\lambda \in W_B(A^*B)} \| \| B_1 \| A_1 + \frac{\bar{\lambda}}{\| B_1 \|} A_2 \|.$$

King'ang'i (2017) also determined the conditions under which the norm of an elementary operator of length two is expressible in terms of the norms of its coefficients operators by proving Corollary 2.3;

Corollary 2.3: (King'ang'i, 2017).

Let Hbe a complex Hilbert space and A_i, B_ibe bounded linear operators on $H$for $i = 1, 2$. Let $0 \in W_{B_1}(B_1^* B_2) \cup W_{B_2}(B_1^* B_2)$. Then $\| E_2 \| \geq \| A_1 \| \| B_1 \|$, where E_2isthe elementary operator of length two.

Theorem 2.4: (King'ang'i, 2017).

Let H be a complex Hilbert space and A_i, B_ibe bounded linear operators on$H$for i= 1, 2. Let E_2be an elementary operator of length two. If $\| A_1 \| \| A_2 \| \in W_{A_1}(A_2 A_1^*)and\| B_1 \| \| B_2 \| \in W_{B_2}(B_1^* B_2)$then

$$\| E_2 \| = \sum_{i=1}^{2} \| A_i \| \| B_i \|.$$

Kawiraet al. (2018) extended the work of King'ang'iet al. (2014) to finite length and determined the norm of an elementary operator of an arbitrary length in a C*-algebra using finite rank operators and proved theorem 2.5;

Theorem 2.5: (Kawiraet al., 2018).

Let Hbe complex Hilbert spaces and $B(H)$ be the algebra of bounded linear operators on H. Let $E_n$be elementary operator on $B(H)$.If $\forall X \in B(H)$ with $\| X \| = 1$, we have $X(f) = f$for all unit vector f ∈H then;

$$\| E_n \| = \textstyle\sum_{i=1}^{n} \| A_i \| \| B_i \|, n \in \mathbb{N}.$$

Muiruri et al. (2018) used the principles of tensor products and the finite rank operator to find the norm of the basic elementary operator in a tensor product of C*-algebras and established the following theorem 2.6:

Theorem 2.6: (Muiruriet al., 2018).

Let $H$and Kbe complex Hilbert spaces and $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$. Then $\forall X \otimes Y \in B(H \otimes K)$ with $\| X \otimes Y \| = 1$, we have $\| M_{A\otimes B, C\otimes D} \| = \| A \| \| B \| \| C \| \| D \|$, where $A, C$and $B, D$are fixed elements in $B(H)$ and $B(K)$ respectively.

Due to the aforementioned, Muiruri et al. (2018) determined the relationship between the norm of basic elementary operator in C*-algebra with the norm of the basic elementary operator in tensor product of C*-algebras, arriving to the conclusion 2.7 below;

Corollary 2.7: (Muiruriet al., 2018).

Let $H$and Kbe complex Hilbert spaces and $B(H \otimes K)$ be the set of bounded linear operators on $\otimes K$. Then Then
$\forall X \otimes Y \in B(H \otimes K)$ with $\| X \otimes Y \| = 1$, we have $\| M_{A\otimes B, C\otimes D} \| = \| M_{A,C} \| \| M_{B,D} \|$, where $M_{A,C}$and $M_{B,D}$ are basic elementary operators in $B(H)$ and $B(K)$ respectively.

Subsequently, Daniel et al. (2022) got the following theorem 2.8 by determining the norm of the basic elementary operator in a tensor product using the concept of Stampli maximal numerical range;

Theorem 2.8: (Daniel et al., 2022).

Let $H$and Kbe Hilbert spaces and let $M_{A\otimes B, C\otimes D}$be basic elementary operator on $B(H \otimes K)$ the set of complex Hilbert space $H \otimes K$.If $\forall U \otimes V \in B(H \otimes K)$ with $\| U \otimes V \| = 1, A, C \in B(H)$,$B, D \in B(K) \zeta \in W_0(C), \xi \in W_0(D)$ then we have $\| M_{A\otimes B, C\otimes D} \backslash B(H \otimes K) \| = Sup_{\zeta \in W_0(C)} Sup_{\xi \in W_0(D)} \{ |\zeta| |\xi| \| A \| \| B \| \}$

Finally, Daniel et al., (2023) determined the bounds of the norm of elementary operator of length two in tensor product using the Stampli's maximal numerical range and obtained theorem 2.9;

Theorem 2.9: (Daniel et al., (2023).

Let $H$and Kbe Hilbert spaces and let $M_{2A\otimes B,C\otimes D}$be basic elementary operator on $B(H \otimes K)$ the set of complex Hilbert space $H \otimes K$.If $\forall U \otimes V \in B(H \otimes K)$ with $\| U \otimes V \|= 1$, $A_i, C_i \in B(H)$,$B_i, D_i \in B(K)\zeta_i \in W_0(C_i), \xi_i \in W_0(D_i)$ then we have $\| M_{2A\otimes B,C\otimes D} \backslash B(H \otimes K) \|= Sup_{\zeta_i \in W_0(C_i)} Sup_{\xi_i \in W_0(D_i)} \{|\zeta_i||\xi_i| \| A_i \|\| B_i \|\}$

Muiruri et al. (2024) investigated the bounds of the norm of an elementary operator of finite length in a tensor product of C*-algebras using the concept of finite rank operator and properties of tensor product of C*-algebras and obtained the theorem 2.10.

Theorem 2.10 (Muiruri et al., 2024)

If H and $K$are complex Hilbert spaces and $B(H \otimes K)$, the set of bounded linear operator on $H \otimes K$. If $\forall X \otimes Y \in B(H\otimes K)$ and $\| X \otimes Y \|= 1$ then;

$$\| T_n \|= \sum_{i=1}^{n} \| A_i \|\| B_i \|\| C_i \|\| D_i \|$$

, where T_n is the Elementary operator of finite length as defined earlier and $A_i, C_i \in B(H)$ and $B_i, D_i \in B(K)$.

Muiruri et al. (2024) also determine the bounds of the norm of Jordan elementary operator in tensor product of C*-algebras and obtained that:-

Theorem 2.11(Muiruri et al., 2024)

Let $H \otimes K$ be tensor product of Hilbert spaces$H$ and $K$and$B(H \otimes K)$ be the set of bounded linear operator on $H \otimes K$. Then$\forall X \otimes Y \in B(H\otimes K)$ with$\| X \otimes Y \|= 1$ we have;

$$\| U_{A\otimes B,C\otimes D} \|= 2 \| A \|\| B \|\| C \|\| D \|$$

, where $U_{A\otimes B,C\otimes D}$ is the Jordan Elementary operator as defined earlier and $A, C \in B(H)$ and $B, D \in B(K)$.

Now, as our main result, we investigate the bounds of the norm of an elementary operator of length two in a tensor product of C∗-algebras using the concept of finite rank operator and properties of tensor product of C*-algebras.

Theorem 1.12

Let $H \otimes K$ be tensor product of Hilbert spaces$H$ and $K$and$B(H \otimes K)$ be the set of bounded linear operator on $H \otimes K$. Then $\forall X \otimes Y \in B(H\otimes K)$ with$\| X \otimes Y \|= 1$ we have;

$$\| T_2 \|= \sum_{i=1}^{2} \| A_i \|\| B_i \|\| C_i \|\| D_i \|$$

, where T_2 is the Elementary operator of finite length as defined earlier and $A_i, C_i \in B(H)$ and $B_i, D_i \in B(K)$.

Proof

By definition $\| T_2 \backslash B(H \otimes K) \|= \sup\{\| T_2(X \otimes Y) \|\}$, $\forall X \otimes Y \in B(H \otimes K), \| X \otimes Y \|= 1$

$\| T_2 \backslash B(H \otimes K) \| \geq \| T_2(X \otimes Y) \|, \quad \forall X \otimes Y \in B(H \otimes K), \| X \otimes Y \|= 1$

Then we have, $\forall \varepsilon \geq 0$

$\| T_2 \backslash B(H \otimes K) \| - \varepsilon < \| T_2(X \otimes Y) \| \ \forall X \otimes Y \in B(H \otimes K), \| X \otimes Y \|= 1$

$$\| T_2\backslash B(H \otimes K) \| -\varepsilon <\| \sum_{i=1}^{2} A_i \otimes B_i (\mathrm{X} \otimes \mathrm{Y}) \mathrm{C}_i \otimes \mathrm{D}_i \| = \| A_1 \otimes B_1 (\mathrm{X} \otimes \mathrm{Y}) \mathrm{C}_1 \otimes \mathrm{D}_1 + A_2 \otimes B_2 (\mathrm{X} \otimes \mathrm{Y}) \mathrm{C}_2 \otimes \mathrm{D}_2 \|$$

Using the property of tensor product of operators, $A_i \otimes B_i (\mathrm{X} \otimes \mathrm{Y}) = A_i X \otimes B_i Y$, then;

$$\| T_2\backslash B(H \otimes K) \| -\varepsilon <\| A_1 X C_1 \otimes B_1 Y D_1 + A_2 X C_2 \otimes B_2 Y D_2 \|$$

Therefore by triangular inequality;

$$\| T_2\backslash B(H \otimes K) \| -\varepsilon <\| A_1 X C_1 \otimes B_1 Y D_1 \| + \| A_2 X C_2 \otimes B_2 Y D_2 \|$$

Clearly, using the tensor product property that $\| A_i X C_i \otimes B_i Y D_i \| = \| A_i X C_i \| \| B_i Y D_i \|$

Then;

$$\| T_2\backslash B(H \otimes K) \| -\varepsilon <\| A_1 X C_1 \| \| B_1 Y D_1 \| + \| A_2 X C_2 \| \| B_2 Y D_2 \|$$

Since $\varepsilon \geq 0$ was arbitrarily taken then

$$\| T_2\backslash B(H \otimes K) \| \leq \| A_1 X C_1 \| \| B_1 Y D_1 \| + \| A_2 X C_2 \| \| B_2 Y D_2 \| \quad (2.1)$$

Since: $\| A_i \mathrm{X} \mathrm{C}_i \| \leq \| \mathrm{A}_i \| \| \mathrm{X} \| \| \mathrm{C}_i \| = \| \mathrm{A}_i \| \| \mathrm{C}_i \|$ since $\| \mathrm{X} \| = 1$

Thus;

$$\| A_i \mathrm{X} \mathrm{C}_i \| \leq \| \mathrm{A}_i \| \| \mathrm{C}_i \|$$

Also $\| B_i \mathrm{Y} \mathrm{D}_i \| \leq \| \mathrm{B}_i \| \| \mathrm{Y} \| \| \mathrm{D}_i \| = \| \mathrm{B}_i \| \| \mathrm{D}_i \|$ since $\| \mathrm{Y} \| = 1$

Then;

$$\| B_i \mathrm{Y} \mathrm{D}_i \| \leq \| \mathrm{B}_i \| \| \mathrm{D}_i \|$$

Thus equation 2.1 becomes

$$\| T_2 \| \leq \| A_1 \| \| B_1 \| \| C_1 \| \| D_1 \| + \| A_2 \| \| B_2 \| \| C_2 \| \| D_2 \|$$

This can be written simply as;

$$\| T_2 \| \leq \sum_{i=1}^{2} \| A_i \| \| B_i \| \| C_i \| \| D_i \| \quad (2.2)$$

Conversely, let there exist unit vector $(e \otimes f)$ of $H \otimes K$ where $e \in H$ and $f \in K$ then

$$\| T_2 (X \otimes Y)(e \otimes f) \| \leq \| T_2 (X \otimes Y) \| \| (e \otimes f) \| \leq \| T_2 \| \| (X \otimes Y) \| \| (e \otimes f) \| = \| T_2 \| \| X \| \| Y \| \| e \| \| f \| = \| T_2 \|$$

Reversing the above equation becomes;

$$\| T_2 \| \geq \| T_2 (X \otimes Y)(e \otimes f) \| = \| \{A_1 \otimes B_1 (X \otimes Y) C_1 \otimes D_1 + A_2 \otimes B_2 (X \otimes Y) C_2 \otimes D_2\}(e \otimes f) \|$$

$$= \| A_1 \otimes B_1 (X \otimes Y) C_1 \otimes D_1 (e \otimes f) + A_2 \otimes B_2 (X \otimes Y) C_2 \otimes D_2 (e \otimes f) \| = \| A_1 X C_1 e \otimes B_1 Y D_1 f + A_2 X C_2 e \otimes B_2 Y D_2 f \|$$

Therefore;

$$\| T_2 \| \geq \| A_1 X C_1 e \otimes B_1 Y D_1 f + A_2 X C_2 e \otimes B_2 Y D_2 f \| \quad (2.3)$$

Thus if we square both sides of equation (2.3):

$$\| T_2 \|^2 \geq \| A_1 X C_1 e \otimes B_1 Y D_1 f + A_2 X C_2 e \otimes B_2 Y D_2 f \|^2$$

$$= \langle A_1 X C_1 e \otimes B_1 Y D_1 f + A_2 X C_2 e \otimes B_2 Y D_2 f, A_1 X C_1 e \otimes B_1 Y D_1 f + A_2 X C_2 e \otimes B_2 Y D_2 f \rangle$$

$$= \langle A_1 X C_1 e \otimes B_1 Y D_1 f, A_1 X C_1 e \otimes B_1 Y D_1 f + A_2 X C_2 e \otimes B_2 Y D_2 f \rangle + \langle A_2 X C_2 e \otimes B_2 Y D_2 f, A_1 X C_1 e \otimes B_1 Y D_1 f + A_2 X C_2 e \otimes B_2 Y D_2 f \rangle$$

$$= \langle A_1 X C_1 e \otimes B_1 Y D_1 f, A_1 X C_1 e \otimes B_1 Y D_1 f \rangle + \langle A_1 X C_1 e \otimes B_1 Y D_1 f, A_2 X C_2 e \otimes B_2 Y D_2 f \rangle + \langle A_2 X C_2 e \otimes B_2 Y D_2 f, A_1 X C_1 e \otimes B_1 Y D_1 f \rangle$$

$$+ \langle A_2 X C_2 e \otimes B_2 Y D_2 f, A_2 X C_2 e \otimes B_2 Y D_2 f \rangle$$

Since $\langle X_1 \otimes Y_1, X_2 \otimes Y_2 \rangle = \langle X_1, X_2 \rangle \langle Y_1, Y_2 \rangle$then:

$= \langle A_1XC_1e, A_1XC_1e\rangle\langle B_1YD_1f, B_1YD_1f\rangle + \langle A_1XC_1e, A_2XC_2e\rangle\langle B_1YD_1f, B_2YD_2f\rangle + \langle A_2XC_2e, A_1XC_1e\rangle\langle B_2YD_2f, B_1YD_1f\rangle$

$+\langle A_2XC_2e, A_2XC_2e\rangle\langle B_2YD_2f, B_2YD_2f\rangle$

$=\| A_1XC_1e \|^2 \| B_1YD_1f \|^2 + \langle A_1XC_1e, A_1XC_1e\rangle\langle B_1YD_1f, B_2YD_2f\rangle + \langle A_2XC_2e, A_1XC_1e\rangle\langle B_2YD_2f, B_1YD_1f\rangle$

$+\| A_2XC_2e \|^2 \| B_2YD_2f \|^2$(2.4)

Now, let $u_i, v_i: H \longrightarrow \mathbb{R}^+$ be functionals for $i = 1,2$

Choose vectors $y, z \in H$ and define finite rank operators $A_i = u_i \otimes y$ and $C_i = v_i \otimes z$ on H for $i = 1,2$by

$A_ie = (u_i \otimes y)e = u_i(e)y \ \forall\ e \in H$ with $\| y \| = 1 i = 1,2$and $C_ie = (v_i \otimes z)e = v_i(e)z$ with $\| z \| = 1 i = 1,2$

Observe that the norm of A_i for $i = 1,2$is:

$$\| A_i \| = \sup\{\| (u_i \otimes y)\, e \| : e \in H, \| y \| = 1\}$$

$$= \sup\{\| u_i(e)y \| : e \in H, \| y \| = 1\}$$

$$= \sup\{|u_i(e)|\ \| y \| : e \in H, \| y \| = 1\}$$

$$= \sup\{|u_i(e)| : e \in H\} = |u_i(e)|$$

That is $\| A_i \| = |u_i(e)| \forall\ e \in H$ with $i = 1,2$

Likewise, the norm of C_i is $\| C_i \| = |v_i(e)| \forall\ e \in H$ with $i = 1,2$

From Equation 2.4 above then:

$$\| A_1XC_1e \|^2 = \| (u_1 \otimes y)X(v_1 \otimes z)e \|^2$$

$$= \| (u_1 \otimes y)Xv_1(e)z \|^2$$

$$= \| v_1(e)(u_1 \otimes y)X(z) \|^2$$

$$= |v_1(e)|^2 \ \| (u_1 \otimes y)X(z) \|^2$$

$$= |v_1(e)|^2 \ \| u_1(X(z))y \|^2$$

$$= |v_1(e)|^2 |u_1(X(z))|^2 \ \| y \|^2$$

$$= |v_1(e)|^2 |u_1(X(z))|^2 = \| A_1 \|^2 \| C_1 \|^2$$

Thus $$\| A_1XC_1e \|^2 = \| A_1 \|^2 \| C_1 \|^2 \quad (2.5)$$

Thus using the same concept also:

$\| B_1YD_1f \|^2 = \| B_1 \|^2 \| D_1 \|^2$ (2.6)

$\| A_2XC_2e \|^2 = \| A_2 \|^2 \| C_2 \|^2$ (2.7)

$\| B_2YD_2f \|^2 = \| B_2 \|^2 \| D_2 \|^2$ (2.8)

Also: $\langle A_1XC_1e, A_2XC_2e\rangle = \langle (u_1 \otimes y)X(v_1 \otimes z)e, (u_2 \otimes y)X(v_2 \otimes z)e\rangle$

$$= \langle (u_1 \otimes y)Xv_1(e)z, (u_2 \otimes y)Xv_2(e)z\rangle$$

$$= \langle v_1(e)(u_1 \otimes y)Xz, v_2(e)(u_2 \otimes y)Xz \rangle$$

$$= \langle v_1(e)u_1(X(z))y, v_2(e)u_2(X(z))y \rangle$$

$$= v_1(e)u_1(X(z))v_2(e)u_2(X(z))\langle y, y \rangle$$

$$= v_1(e)u_1(X(z))v_2(e)u_2(X(z))$$

Since $v_1(e), u_1(X(z)), v_2(e)$ and $u_2(X(z))$ are all positive real numbers, we have:

$v_1(e) = |v_1(e)| = \| C_1 \|, u_1(X(z)) = |u_1(X(z))| = \| A_1 \|$, $v_2(e) = |v_2(e)| = \| C_2 \|$ and

$$u_2(X(z)) = |u_2(X(z))| = \| A_2 \|$$

Thus we have $\langle A_1XC_1e, A_2XC_2e \rangle = v_1(e)u_1(X(z))v_2(e)u_2(X(z)) = \| C_1 \| \| A_1 \| \| C_2 \| \| A_2 \|$

Since the norms of A_i and C_i for $i = 1,2$are scalars then :

$\langle A_1XC_1e, A_2XC_2e \rangle = \| A_1 \| \| A_2 \| \| C_1 \| \| C_2 \|$(2.9)

Hence using the same concept as above then:

$\langle B_1YD_1f, B_2YD_2f \rangle = \| B_1 \| \| B_2 \| \| D_1 \| \| D_2 \|$ (2.10)

It then follows that $\langle A_2XC_2e, A_1XC_1e \rangle = \langle (u_2 \otimes y)X(v_2 \otimes z)e, (u_1 \otimes y)X(v_1 \otimes z)e \rangle$

$$= \langle (u_2 \otimes y)Xv_2(e)z, (u_1 \otimes y)Xv_1(e)z \rangle$$

$$= \langle v_2(e)(u_2 \otimes y)Xz, v_1(e)(u_1 \otimes y)Xz \rangle$$

$$= \langle v_2(e)u_2(X(z))y, v_1(e)u_1(X(z))y \rangle$$

$$= v_2(e)u_2(X(z))v_1(e)u_1(X(z))\langle y, y \rangle$$

$$= v_2(e)u_2(X(z))v_1(e)u_1(X(z)) = \| C_2 \| \| A_2 \| \| C_1 \| \| A_1 \|$$

Since the norms of A_i and C_i for $i = 1,2$are scalars then :

$\langle A_2XC_2e, A_1XC_1e \rangle = \| A_1 \| \| A_2 \| \| C_1 \| \| C_2 \|$ (2.11)

Thus using the same concept then:

$\langle B_2YD_2f, B_1YD_1f \rangle = \| B_1 \| \| B_2 \| \| D_1 \| \| D_2 \|$ (2.12)

Thus substituting equations 2.5 to 2.12 in 2.4 then

$\| T_n \|^2 \geq \| A_1 \|^2 \| B_1 \|^2 \| C_1 \|^2 \| D_1 \|^2 + \| A_1 \| \| B_1 \| \| C_1 \| \| D_1 \| \| A_2 \| \| B_2 \| \| C_2 \| \| D_2 \| +$

$\| A_1 \| \| B_1 \| \| C_1 \| \| D_1 \| \| A_2 \| \| B_2 \| \| C_2 \| \| D_2 \| + \| A_2 \|^2 \| B_2 \|^2 \| C_2 \|^2 \| D_2 \|^2$

$\| T_n \|^2 \geq \| A_1 \|^2 \| B_1 \|^2 \| C_1 \|^2 \| D_1 \|^2 + 2 \| A_1 \| \| B_1 \| \| C_1 \| \| D_1 \| \| A_2 \| \| B_2 \| \| C_2 \| \| D_2 \| + \| A_2 \|^2 \| B_2 \|^2 \| C_2 \|^2 \| D_2 \|^2$

This implies that:

$\| T_2 \|^2 \geq \{ A_1 \| \| B_1 \| \| C_1 \| \| D_1 \| + \| A_2 \| \| B_2 \| \| C_2 \| \| D_2 \| \}^2$

Thus obtaining square root in both sides:

$$\| T_2 \| \geq \| A_1 \| \| B_1 \| \| C_1 \| \| D_1 \| + \| A_2 \| \| B_2 \| \| C_2 \| \| D_2 \|$$

Finally, it's clear that:

$$\parallel T_2 \parallel \geq \sum_{i=1}^{2} \parallel A_i \parallel\parallel B_i \parallel\parallel C_i \parallel\parallel D_i \parallel (2.13)$$

From (2.2) and (2.13) then

$$\parallel T_2 \backslash B(H\otimes K) \parallel = \sum_{i=1}^{2} \parallel A_i \parallel\parallel B_i \parallel\parallel C_i \parallel\parallel D_i \parallel$$

CONCLUSION

From the main result the paper have extended the definition of elementary operator in tensor product of C*-algebras to other examples of elementary operators and determined the bounds of the norm of elementary operator of length two in tensor product of C*-algebras. The norm of other types of elementary operator like left and right multiplication operators and inner and general derivation operators in tensor product of C*-algebras can also be determined.

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