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THE FASCINATING MATHEMATICAL BEAUTY OF THE SPECIAL MATRIX BASED ON INFINITE CONVERGENT GEOMETRIC SERIES

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ABSTRACT

The infinite Geometric Series is a series of the form

$\sum_{k=0}^{\infty} ax^k$, where a is aconstant. The geometric power series $\sum_{k=0}^{\infty} ax^k$ converges for $|x|<1$ and is equal to $\frac{a}{1-x}$. The Second Derivative of $\sum_{k=0}^{\infty} ax^k$ is $\sum_{k=2}^{\infty} ak(k-1)x^{k-2} = \sum_{k=0}^{\infty} a(k+2)(k+1)x^k$

Let t be sequence in (0,1) that converges to 1. The matrix based on second derivative of convergent infinite geometric series defined as

$a_{nk}=\frac{1}{2}(k+2)(k+1)(1-t_n)^3 t_n{}^k$. We denote this matrix by S_t and name it the matrix associated second derivative of geometric series. S_t is a sequence to sequence mapping. When a matrix S_t is applied to a sequence x, we get a new sequence $S_t x$ whose nth term is given by:

$$(S_t x)_n = \frac{1}{2}(1-t_n)^3 \sum_{k=o}^{\infty} (k+2)(k+1)t_n{}^k x_k$$

The sequence $S_t x$is called the S_t-transform of the sequence x.

The purpose of this research is to investigate the effect of applying S_t to convergent sequences, bounded sequences, divergent sequences, and absolutely convergent sequences. We considering and answer the following interesting main research questions.

KEYWORDS

Fibonacci numbers, Fibonacci sequences, Pascal's triangle, and Golden ratio.

Research Questions

(1) What is the domain of t for which S_t maps convergent sequence into convergent sequence?

(2) What is the domain of t for which the S_t maps absolutely convergent sequence into absolutely convergent sequence?

(3) Does S_t maps unbounded sequence to convergent sequence?

(4) Does S_t maps divergent sequence to convergent sequence?

(5) How is the strength of the S_t comparing to the identity matrix?

Notations and Background Materials

w= {the set of all complex sequences}

c= {the set of all convergent complex sequences}

$c(A)$ ={y: Ay$\in$ c}

l = {y: $\sum_{k=0}^{\infty}|y_k| < \infty$}

$l(A)$ ={y: Ay$\in$ l }

Definition 1: A matrix A is an x-y matrix if the image Au of u under the transformation A is in Y wherever u is in x.

Regular Matrix

A matrix is regular if $lim_{n\to\infty}Z_n$= a$\Rightarrow$ $lim_{n\to\infty}(AX)_n$=a. That is a sequence Z is convergent to A$\Rightarrow$ the A-transform of Z also converses to a.

The Sliverman-Toeplitz Rule

We state the following famous Sliverman-Toeplitz Rule as Proposition I with out proof and apply it.

Proposition I: A matrix A $=(a_{n,k})$ is regular if and only if

(i) $\lim_{n\to\infty} a_{n,k} = 0$ for each k= 0,1,…,

(ii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} = 1,$ and

(iii) $\sup_n \{ \sum_{k=0}^{\infty} |a_{n;k}| \} \leq M < \infty$ for some $M < 0.$

The Main Results

Theorem 1: The S_t matrix is a regular matrix for all t.

Proof: We use proposition 1, to prove the theorem. Note that

(1) $\lim_{n\to\infty} a_{n,k} = \lim_{n\to\infty} \frac{1}{2}(k+2)(k+1)(1-t_n)^3 t_n^{\ k} = 0$

(2) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \lim_{n\to\infty} \frac{1}{2}\sum_{k=0}^{\infty}(k+2)(k+1)t_n^{\ k}(1-t_n)^3 =$

$\lim_{n\leftarrow\infty} \frac{1}{2}(1-t_n)^3 \sum_{k=0}^{\infty}(k+2)(k+1)t_n^{\ k} = \frac{(1-t_n)^3}{(1-t_{n)})^3} = 1$ and

(3) $Sup_n \sum_{k=0}^{\infty} a_{n,k} = 1$

Hence by Proposition I, the matrix S_t is <u>a regular matrix</u>. Thus the matrix S_t maps all convergent sequences into convergent sequences and we can say that the matrix S_t a c-c matrix.

Remark 1: The S_t matrix maps <u>a bounded sequence</u> into a convergent sequence as shown by the following example. This shows that the S_t matrix is stronger than the identity matrix or c(S_t) is larger than c.

Example1: Consider the bounded sequence given by $x_k = (-1)^k$

Then $(S_t x)_n = \frac{1}{2}(1-t_n)^3 \sum_{k=0}^{\infty}(k+2)(k+1)(t_n)^k(-1)^k$

$$= \frac{1}{2}(1-t_n)^3 \sum_{k=0}^{\infty}(k+2)(k+1)(-t^n)^k$$

$$= (1-t_n)^3 \frac{1}{(1+t_n)^3}$$

$$\Rightarrow (S_t x)_n = \frac{(1-t_n)^3}{(1+t_n)^3} \Rightarrow \lim_{n\to\infty}(S_t x)_n = 0 \Rightarrow S_t x \in c$$

Remark 2: The S_t matrix maps also a divergent sequence x into a convergent sequence as shown by the following example.

Example 2: Consider the unbounded sequence given by x defined by

$x_k = (-1)^k (k+3)(k+1)(k+2)$. Note that

$$(S_t x)_n = \frac{1}{2} \sum_{k=0}^{\infty} (1-t_n)^3 t_n^k (-1)^k (k+2)(k+1)$$

$$= \frac{1}{2}(1-t_n)^3 \sum_{k=0}^{\infty} t_n^k (-1)^k)(k+3)(k+2)(k+1)$$

$$= (1-t_n)^3 \sum_{k=0}^{\infty} (-t_n)^k (k+3)(k+2)(k+1)$$

$$= \frac{3(1-t_n)^3}{(1+t_n)^4}$$

Now , $\lim_{n\to\infty} (S_t x)_n = \lim_{n\to\infty} \frac{(1-t_n)^3}{(1+t_n)^4} = 0$

Hence $S_t x \in c.$

Knopp-LorentzThorem

The Matrix A is an ℓ - ℓ matrix if and only if there exists a number $M > 0$ such that for every $k,$

$$\sum_{n=0}^{\infty} |a_{nk}| \leq M.$$

Theorem 2: S_t is ℓ - $\ell \Leftrightarrow (1-t)^3 \in \ell$

Lemma 1:

$$S_t \text{ is } \ell - \ell_{\text{matrix}} \Rightarrow (1-t)^3 \in \ell \,.$$

Proof: We use the Knopp-Lorentz Rule.

$$S_t \text{ is } \ell - \ell \Rightarrow \sum_{n=0}^{\infty} \left|(1-t_n)^3 t_n^k\right| \leq M$$

$$\Rightarrow \sum_{n=0}^{\infty} \left|(1-t_n)^3\right| \leq M \quad \text{(for k=0)}$$

$$\Rightarrow (1-t)^3 \in \ell$$

Lemma 2:

$$(1-t)^3 \in \ell \Rightarrow S_t \text{ is an } \ell - \ell_{\text{matrix}}$$

Proof: We use the Knopp-Lorentz Rule

$$\sum_{n=0}^{\infty} |a_{nk}| \leq \sum_{n=0}^{\infty} \left|(1-t_n)^3 t_n^k\right|$$

$$\leq \sum_{n=0}^{\infty} (1-t_n)^3 \leq M \text{ for some M>0 as } (1-t)^3 \in \ell \,.$$

Now Theorem 2 follows by Lemmas 1&2.

Corollary 1. $\arcsin(1-t)^2 \in 1 \Leftrightarrow S_t$ is an l-l matrix.

Proof: The corollary easily follows using Theorem 2 and the following basic inequality.

$$(1\text{-t})^3 \leq \arcsin(1\text{-t})^3 \leq \frac{(1-t)^3}{\sqrt{1-(1-t)^3}} \,.$$

Theorem 3 $\frac{-1}{\ln(1-t_n)} \in 1 \Rightarrow \mathbf{S}_t$ is an l-l matrix.

Proof. Note that:

$$(1-t_n)^3 \le \text{(1-t}_n\text{):=}\left(\sum_{k=0}^{\infty} t_n{}^k\right)^{-1}$$

$$\le \left(\sum_{k=0}^{\infty} \frac{1}{k+1} t_n{}^k\right)^{-1}$$

$$= \left(\sum_{k=0}^{\infty} t_n{}^k \left(\int_0^1 V^k dV\right)\right)^{-1}$$

$$= \left(\sum_{k=0}^{\infty} \left(\int_0^1 t_n{}^k V^k dV\right)\right)^{-1}$$

$$= \left(\int_0^1 dV \left(\sum_{k=0}^{\infty} (t_n V)^k\right)\right)^{-1}$$

The Interchanging of the Integral and summation is legitimate as the power series

$\sum_{k=0}^{\infty} (Vt_n)^k$ converges absolutely and uniformly for $0 \le Vt_n \le 1$. Hence we have,

$$(1-t_n)^3 \le \text{1-t}_n \le \left(\int_0^1 \frac{dV}{1-Vt_n}\right)^{-1}$$

$$= \left(\frac{-1}{t_n}(\ln(1-t_n)\right)^{-1}$$

$$\le \frac{-1}{\ln(1-t_n)}$$

The hypothesis that $\frac{-1}{\ln(1-t)} \in \text{l} \Rightarrow \text{(1-t)}^3 \in \text{l}$ and hence by Theorem 2,

S_t is l-l.

Remark 3. An l-l S_t matrix maps a bounded sequence into l as shown by the following example. This shows that the S_t matrix is stronger than the identity matrix in the *l-l* setting or l(S_t) is larger than *l*.

Example 3.

Assume the S_t matrix is *l-l* and consider the bounded sequence given by $x_k = (-1)^k$

Then $(S_t x)_n = \frac{1}{2}(1-t_n)^3 \sum_{k=0}^{\infty}(k+1)(k+2)(t_n)^k(-1)^k$

$$= \frac{1}{2}(1-t_n)^3 \sum_{k=0}^{\infty}(k+2)(k+1)(-t^n)^k$$

$$= (1-t_n)^3 \frac{1}{(1+t_n)^3}$$

$$\leq (1-t_n)^3$$

Now the S_t matrix is l-$l \Rightarrow$ $(1-t)^3 \in$ l ,by Theorem 2, and hence $S_t x \in l.$

Remark 4: An l-l S_t matrix maps unbounded sequence into l as shown by the following example.

Example 4: Assume S_t is an *l-l* matrix and consider the unbounded sequence given by

$x_k = (-1)^k (k+3)$. Note that

$$(S_t x)_n = = \frac{1}{2}(1-t_n)^3 \sum_{k=0}^{\infty} t_n^k (-1)^k (k+3)(k+2)(k+1)$$

$$= \frac{1}{2}(1-t_n)^3 \sum_{k=0}^{\infty} (-t_n)^k (k+3)(k+2)(k+1)$$

$$= \frac{(1-t_n)^3}{(1+t_n)^4}$$

$$\leq (1-t_n)^3$$

Now S_t is an is l-*l matrix* $\Rightarrow$ $(1-t)^3 \in$ l , by Theorem 2, and hence $S_t x \in l.$

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